# Berkeley Math Circle: Monthly Contest 5 Solutions

1. How many ways are there to color the vertices of an equilateral triangle red, green, or blue, where colorings that can be obtained from each other by rotation or reflection are considered the same?

   *Solution.* We use casework. If the vertices are all the same color, there are 3 distinct colorings, one for each color. If two are one color and one is a third color, there are 3 ways to choose the first color and 2 to choose the second, giving $3 \cdot 2 = 6$. If they are all different colors, there is only one coloring up to rotation and reflection. This gives a total of $3 + 6 + 1 = 10$. □

2. Let $\left\{ {n \atop k} \right\}$ be the number of ways to partition $n$ objects into $k$ nonempty subsets. For example, $\left\{ {4 \atop 2} \right\} = 7$ since the four objects $A$, $B$, $C$, $D$ can be partitioned in exactly seven ways: $\{A, BCD\}$, $\{B, CDA\}$, $\{C, DAB\}$, $\{D, ABC\}$, $\{AB, CD\}$, $\{AC, BD\}$, $\{AD, BC\}$.

   Prove that
   $$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

   *Solution.* Divide the $\left\{ {n \atop k} \right\}$ partitions into two cases: either 1 is alone in its subset, or it is not. If 1 is alone, the remaining $n - 1$ objects must be divided into $k - 1$ nonempty subsets, and $\{1\}$ will be the $k^{\text{th}}$ subset. If 1 is not alone, the remaining objects must be partitioned into $k$ susbets, and there are then $k$ ways to choose which subset 1 is in. □

3. Show that every positive integer $n$ can be uniquely written in the form
   $$n = \sum_{i=0}^{k} d_i \cdot 2^i = d_0 + 2d_1 + \dots 2^k d_k$$
   for some positive integer $k$, where each $d_i$ either 1 or 2.

   *Solution.* Start with the usual binary representation of $n$. If it is all 1's, we are done; otherwise, we can repeatedly apply the transformations $10 \mapsto 02$ (since $2^i = 2 \cdot 2^{i-1}$) and $20 \mapsto 12$ (since $2 \cdot 2^i = 1 \cdot 2^i + 2 \cdot 2^{i-1}$) until the resulting string contains only 1's and 2's. This process must terminate, since the number of non-leading 0's starts finite and is strictly decreasing. Further, we can reverse these transformations to get back to the original binary number, so since binary representations are unique, this representation must also be unique. □

4. A group of prisoners play the following game: each is given a red or black hat, and they stand in a line such that each can see the hats of everyone in front of him, but not his own or those of anyone behind him. Starting from the end of the line, each

in turn must guess his hat color, and they will win the game and all be freed if at most one of them guess incorrectly.

They are allowed to discuss a strategy beforehand, and they can hear the previous guesses, but no other communication is allowed during the game. Show that they can always win.

*Solution.* The prisoner at the end of the line counts the number of red hats he sees in front of him. If this number is even, he says "red," and if it is odd, he says "black." The next prisoner now counts the number of red hats in front of him, and based on its parity can determine the color of his own hat. Continuing in this manner, each prisoner can use the responses of the previous prisoners to determine the color of his own hat, and at most the first prisoner will be wrong. Thus, they will always be able to win the game. □

5. Let $ABC$ be an equilateral triangle. Points $L$, $P$, and $Q$ lie on the segments $AB$, $AC$, and $BC$, respectively, and are such that $PCQL$ is a parallelogram. Let $M$ be the midpoint of $AB$. The circle with center $M$ passing through $C$ intersects the circle with diameter $CL$ again at $T$. Prove that lines $AQ$, $BP$, and $LT$ are concurrent.

   *Solution.* Let $K$ be the reflection of $L$ across $M$. Let $X = \overline{AQ} \cap \overline{BP}$. Then it suffices to show that lines $XL$ and $CK$ are parallel, since $M$ lies on the perpendicular bisector of $CT$, and $\angle CTL = 90^\circ$.

   Note triangles $PAL$ and $BQL$ are equilateral too. We claim that $\overline{XL}$ passes through point $D$, the reflection of $C$ across $\overline{AB}$. This will certainly imply the conclusion.

   Note that $PXLA$ and $QXBL$ are cyclic, since $\triangle PLB \cong \triangle ALQ$ provides the necessary angles. Now, line $XL$ is the radical axis, and $D$ has equal power to both the circumcircles of $PAL$ and $QBL$, as $DA$ and $DB$ are congruent tangents. This completes the proof. □

6. Suppose 4951 points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

   *Solution.* This is an example of a direct greedy algorithm: we will simply grab points until we are stuck.

   Consider a maximal set $S$ of the points as described (meaning no more additional points must be added), and suppose $|S| = n$. Then the $4951 - n$ other points must each lie on a line determined by two points in $S$, meaning

$$4951 - n \le \binom{n}{2} \implies n + \binom{n}{2} \ge 4951.$$

   This requires $n \ge 100$. □

7. For positive real numbers $a$, $b$, $c$ prove that

$$\frac{1+bc}{a} + \frac{1+ca}{b} + \frac{1+ab}{c} > \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

*Solution.* We observe that

$$\begin{aligned}\frac{1}{a}+\frac{1}{2}a\left(\frac{b}{c}+\frac{c}{b}\right) &\geq \frac{1}{a}+\frac{1}{2}a\cdot 2\sqrt{\frac{b}{c}\cdot\frac{c}{b}}\\ &= a+\frac{1}{a}=\sqrt{a^2+2+a^{-2}}\\ &> \sqrt{a^2+2}\end{aligned}$$

with the first inequality by AM-GM. Cyclically summing concludes the proof. □