Berkeley Math Circle: Monthly Contest 3 Solutions

1. Find all ordered pairs (x, y) of positive integers such that $xy - x - y = 11$.

   Solution. Adding 1 to both sides and factoring the left side gives

   $$(x-1)(y-1) = 12.$$

   Using the factors of 12, we find that the possible ordered pairs are

   $$(2,13),\ (3,7),\ (4,5),\ (5,4),\ (7,3),\ (13,2).$$ □

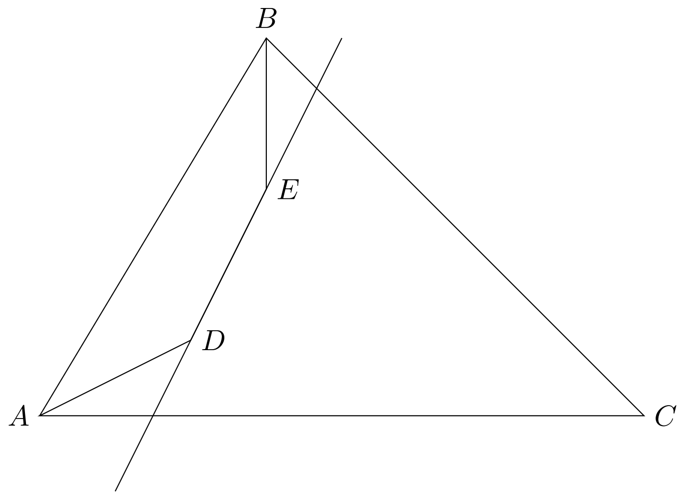
2. How many numbers are there from 1 to 100 that are neither a multiple of 7 nor contain the digit 7?

   Solution. Of the 100 numbers, $\lfloor 100/7 \rfloor = 14$ are multiples of 7, and $9^2 = 81$ do not contain the digit 7 (since there are 9 choices for each of the 2 digits, including 100 and excluding 0), so $100 - 81 = 19$ do. There are 3 numbers that have both (7, 70, and 77). By the principle of inclusion/exclusion, this gives a total of

   $$100 - 14 - 19 + 3 = 70.$$ □

3. Show that for any five points in the plane, no three of which are collinear, some four form a convex quadrilateral.

   Solution. Suppose not. Then three of the points must form a triangle with the other two points inside, WLOG A, B, C, and the other two points D and E.



   Then the line DE cuts the triangle into two pieces, so one of the pieces contains two of the points A, B, C, WLOG A and B. Then the quadrilateral $ABED$ is convex. □

4. In the game *Sprouts*, there are initially n spots drawn on a plane, and on each move two spots are connected with an edge and a new spot is drawn on this edge. No two edges can cross, and no spot may have more than 3 edges coming from it. An edge may be drawn from a spot to itself. The game ends when no more moves can be made.

   a) Show that the game must end in at most $3n - 1$ moves.

   b) Show that the game will last at least $2n$ moves.

Solution. a) We can say that each spot initially has three lives, and that on each move, one life is lost, since the two ends of the new edge each lose a life, and the new spot on this edge has one life. Thus, after $3n - 1$ moves, there is only one life remaining, and so after $3n - 1$ moves, there is only one life remaining, while an edge requires two lives, so no more moves can be made.

b) Say a spot is *alive* if it has at least one life left, and *dead* otherwise. If the game ends after m moves, there will be $3n - m$ lives remaining, and no sport can have two lives left, or an edge could be drawn from that spot to itself, so there are $3n - m$ live spots left.

Also, no two live spots can be adjacent, or another edge could be drawn between them, so each live spot must be connected to two dead spots. Further, each dead spot can be connected to at most one live spot, or else another edge could be drawn, so there are at least twice as many dead spots as live spots. But the total number of spots is $n+m$, since one spot is added each move. Putting this together,

$$3n - m + 2(3n - m) \leq n + m \implies 8n \leq 4m \implies m \geq 2n. \qquad \square$$

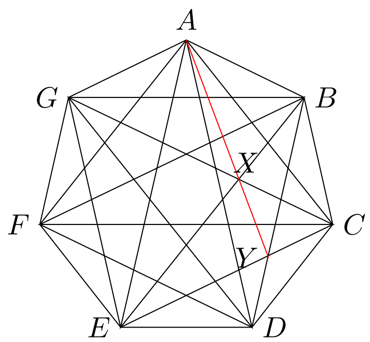
5. Let $ABCDEFG$ be a regular heptagon. Let X be the intersection of diagonals BE and CG, and let Y be the intersection of BD and CE. Prove that A, X, and Y are collinear.

Solution. Note that by AA,

$$\triangle CYD \sim \triangle AG \implies \frac{CY}{CD} = \frac{AG}{GB}.$$

Also by AA,

$$\triangle FGB \sim \triangle CXE \implies \frac{GB}{GF} = \frac{XE}{XC}.$$



Thus

$$\frac{AG}{YC} = \frac{GB}{CD} = \frac{GB}{GF} = \frac{EX}{XC} = \frac{GX}{XC}.$$

Since $\angle AGX = \angle YCX$, we have

$$\triangle AXG \sim \triangle YXC \implies \angle AXG = \angle YXC,$$

which implies that A, X, Y are collinear, as desired. $\square$

6. Let $\phi(n)$ be the number of positive integers less than or equal to n and relatively prime to n. Evaluate

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n}.$$

Solution. We rewrite this as a sum geometric series, which we can then expand:

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n} = \sum_{n=1}^{\infty} \phi(n) \frac{\left(\frac{2}{9}\right)^n}{1 - \left(\frac{2}{9}\right)^n} = \sum_{n=1}^{\infty} \phi(n) \sum_{k=1}^{n} \left(\frac{2}{9}\right)^{kn}.$$

Note that we will get a $\left(\frac{2}{9}\right)^n$ term once for each divisor of n. Thus we can rearrange this sum as

$$\sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n \sum_{k|n} \phi(n).$$

Using that $\sum_{k|n} \phi(k) = n$, this sum becomes

$$\sum_{n=1}^{\infty} n \left(\frac{2}{9}\right)^n = \frac{\frac{2}{9}}{\left(1 - \frac{2}{9}\right)^2} = \frac{18}{49}.$$ $\square$

7. Find all pairs of prime numbers (p, q) such that $p^2 - p - 1 = q^3$.

Solution. Rearranging and factoring gives

$$p^2 - p = p(p-1) = q^3 + 1 = (q+1)(q^2 - q + 1).$$

Since p is a prime, either $p|q+1$ or $p|q^2 - q + 1$. In the former case, $p < q$, so $p - 1 < q$. Since $q^2 - q + 1|p - 1$, this means

$$q^2 - q + 1 < q \implies q^2 - 2q + 1 = (q-1)^2 < 0,$$

which is impossible. Thus $p|q^2 - q + 1$, and so $q + 1|p - 1$. Let $p - 1 = k(q + 1)$. Then

$$q^2 - q + 1 = kp = k(k(q+1)+1) = k^2q + (k^2 + k) \implies q^2 - (k^2+1)q - k^2 - k + 1 = 0.$$

Using the quadratic formula to solve for q gives

$$q = \frac{k^2 + 1 \pm \sqrt{(k^2+1)^2 + 4(k^2 + k - 1)}}{2}.$$

Since q must be an integer, the discriminant must be a perfect square, so

$$(k^2+1)^2 + 4(k^2 + k - 1) = k^4 + 2k^2 + 1 + 4k^2 + 4k - 4 = k^4 + 6k^2 + 4k - 3$$

is a perfect square. Clearly it is more than $(k^2+1)^2$. Now,

$$(k^2+2)^2 = k^4+4k^2+4 = k^4+6k^2+4k-3 \implies 2k^2+4k-7=0.$$

However, this quadratic does not have integer solutions. However,

$$(k^2+3)^2 = k^4+6k^2+9 = k^4+6k^2+4k-3 \implies k=3.$$

This gives

$$q = \frac{10 \pm \sqrt{100+4\cdot 11}}{2} = 11$$

and $p = k(q+1)+1 = 37$.

But

$$(k^2+4)^2 = k^4+8k^4+16 \le k^4+6k^2+4k-3 \implies 2k^3-4k+19 = 2(k-1)^2+17 \le 0,$$

which is impossible. Thus, we cannot have

$$k^4+6k^2+4k-3 = (k^2+n)^2$$

for any $n \geq 4$, so the only possibility is $n=3$. Thus, the only possible solution is $(p,q) = (37,11)$, and we can plug this in to find that it does indeed work. $\square$