Berkeley Math Circle: Monthly Contest 6 Solutions

1. Prove that

$$n(n+1)(2n+1)$$

is always divisible by 6, for n a positive integer.

Solution. The number is even, because either n or $n+1$ is even.

Now we show it is always divisible by three. Assume for contradiction that it isn't. Then neither n nor $n+1$ is divisible by three, so $n+2$ must be. However, $2n+1 = 2(n+2)-3$ is then also a multiple of three, which is a contradiction.

In fact, one can also notice the result from the fact that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

□

2. Oscar draws a triangle ABC on a sheet of paper. He finds that the side lengths of ABC are all powers of 2 (i.e. among 1, 2, 4, 8, ...). Prove that Oscar's triangle is isosceles.

Solution. Consider a longest side of the triangle, 2^a. We claim that another side must have this length too. Otherwise, suppose for contradiction they are 2^b and 2^c where $b, c < a$ Then

$$2^b + 2^c < 2^{a-1} + 2^{a-1} = 2^a$$

which contradicts the triangle inequality.

Hence there must be a second side of length 2^a. □

3. Let a, b, c, d be positive integers such that $ab = cd$. Prove that $a+b+c+d$ is not a prime number.

Solution. Note that

$$\begin{aligned} a(a+b+c+d) &= a^2 + ac + ad + ab \\ &= a^2 + ac + ad + cd \\ &= (a+c)(a+d). \end{aligned}$$

If $a+b+c+d$ was prime it would then have to divide either $a+c$ or $a+d$, which is impossible. □

4. Prove that there exists an infinite sequence of a_1, a_2, ... positive integers such that the following condition holds: $\gcd(a_m, a_n) = 1$ if and only if $|m-n| = 1$.

Solution. Enumerate the primes $p_1, q_1, p_2, q_2, \dots$ and define

$$a_n = p_n q_n \cdot \begin{cases} \prod_{k=1}^{n-2} p_k & n \text{ even} \\ \prod_{k=1}^{n-2} q_k & n \text{ odd.} \end{cases}$$

This works by construction. The idea is that you just take every pair $i < j$ you want to not be relatively prime (meaning $|i - j| \geq 2$) and throw in a prime. You can't do this by using a different prime for every pair (since each a_i must be finite) and you can't use the same prime for a fixed i, so you do the next best thing and alternate using even and odd and you're done. □

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

Solution. Let $[\mathcal{P}]$ denote the area of a polygon $\mathcal{P}$.

The important claim is that if $\overline{KL} \parallel \overline{MN}$, then $[KLM] = [KLN]$. This is a simple consequence of the formula $A = \frac{1}{2}bh$.

Then, we find that

$$\begin{aligned} [ABC] &= [XBC] \quad (\text{since } \overline{AX} \parallel \overline{BC}) \\ &= [XYC] \quad (\text{since } \overline{BY} \parallel \overline{XC}) \\ &= [XYZ] \quad (\text{since } \overline{CZ} \parallel \overline{XY}) \end{aligned}$$

as desired. □

6. A bulldozer is touring Pascal's triangle. It starts at the top of the triangle, at $\binom{0}{0} = 1$. Each move, it travels to an adjacent positive integer, but can never return to a spot it has already visited. Moreover, if it has visited two numbers $a > b$, it may not visit $a + b$ or $a - b$. Finally, the bulldozer is confined to the first 140 rows of Pascal's triangle.

Prove that the bulldozer may visit at least 2017 numbers. (By convention, the nth row contains the entries $\binom{n-1}{k}$ for $k = 0, \dots, n - 1$, hence the nth row has n entries.)

Solution. The main idea is to visit odd numbers!

We claim inductively that the first 2^n rows of Pascal's triangle satisfy the following properties:

- The 2^nth row contains only odd numbers.
- The first 2^n rows contain 3^n odd numbers.
- When taken modulo 2, there is 120 degree symmetry
- There is a path starting at any corner to any other corner through only odd numbers.

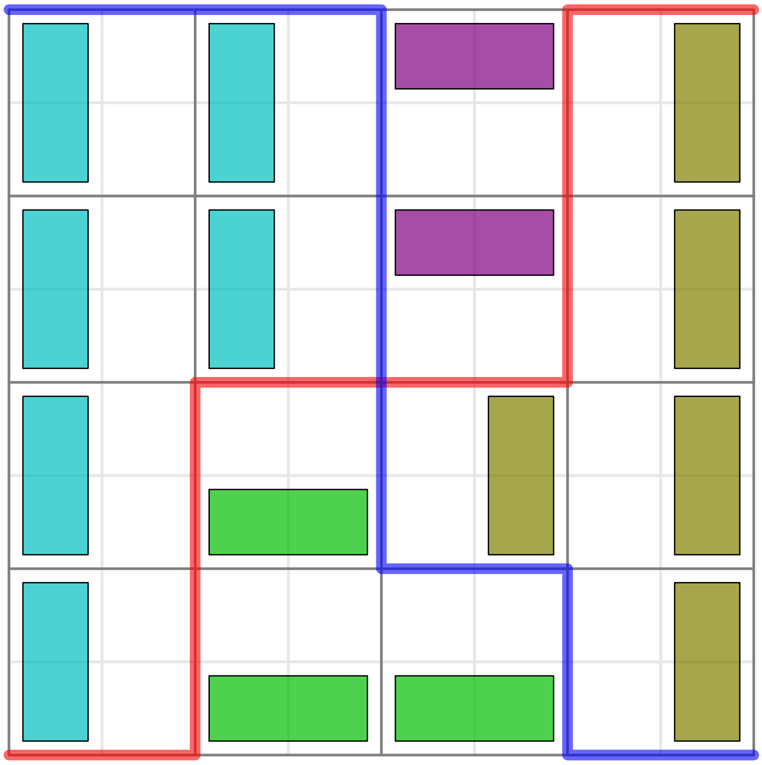
Indeed this is clear for $n = 1$. For the inductive step, let T denote the shape of the first 2^n rows modulo 2. Note that row $2^n + 1$ contains all even numbers except the endpoints $\binom{2^n}{0} = \binom{2^n}{2^n} = 1$. Thus in fact we get two side-by-side copies of the triangle T, which meet on row 2^{n+1} and thus have all ones. (Between the two copies of T we get an inverted triangle having all entries 0.) From these observations, we see that all statements in the inductive hypothesis hold.

Thus, we may visit odd numbers from rows 1 to 128. In doing so, we visit $3^7 = 2187$ odd numbers. $\square$

7. We wish to place ways exactly 100 dominoes (of size 2×1 or 1×2) without overlapping on a 20×20 chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column. In how many ways can this be done?

Solution. The answer is $\binom{20}{10}^2$.

Generalizing the problem slightly, the answer is $\binom{m+n}{n}^2$ for a $2m \times 2n$ rectangle. We provide a "proof without words" with the following bijection:



$\square$