Berkeley Math Circle: Monthly Contest 5 Solutions

1. A bird thinks the number $2n^2 + 29$ is prime for every positive integer n. Find a counterexample to the bird's conjecture.

   Solution. Simply taking $n = 29$ works, since $2 \cdot 29^2 + 29 = 29(2 \cdot 29 + 1) = 29 \cdot 59$. □

2. An iguana writes the number 1 on the blackboard. Every minute afterwards, if the number x is written, the iguana erases it and either writes $\frac{1}{x}$ or $x + 1$. Can the iguana eventually write the number $\frac{20}{17}$?

   Solution. Yes. First, the iguana writes

$$1 \to 2 \to \frac{1}{2} \to \frac{3}{2} \to \frac{2}{3}.$$

   Then, the iguana adds 1 to arrive at $\frac{17}{3}$. Finally, finish with

$$\frac{17}{3} \to \frac{3}{17} \to \frac{20}{17}.$$

   In fact, see if you can prove that *any* positive rational number can be achieved! □

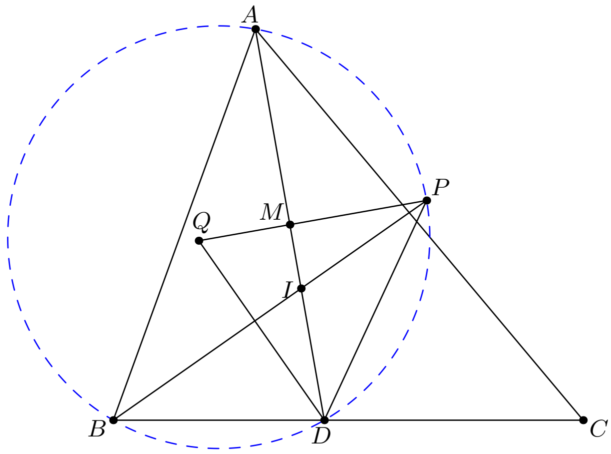
3. We define a *chessboard polygon* to be a polygon whose edges are situated along lines of the form $x = a$ and $y = b$, where a and b are integers. These lines divide the interior into unit squares, which we call cells.

   Let n and k be positive integers. Assume that a square can be partitioned into n congruent chessboard polygons of k cells each. Prove that this square may also be partitioned into k congruent chessboard polygons of n cells each.

   Solution. Note that $nk = s^2$ for some s. By Factor Lemma, pick $n = ab$, $k = cd$, and $s = ac = bd$. Now we can tile the board with $a \times b$ rectangles! □

4. Let ABC be a triangle, I the incenter, and D the intersection of lines AI and BC. The perpendicular bisector of AD meets BI and CI at P and Q. Show that I is the orthocenter of triangle PQD.

   Solution. It suffices to show that $CI \perp PD$.


A
P
M
Q
I
B
D
C

Note that since $AP = PD$ and BI is bisector of $\angle ABD$, point P lies on the circumcircle of $\triangle ABD$ (on the midpoint of the arc). From this one can compute $\angle PDC = \angle IDC - \angle ADP = \angle IDC - \angle ABI$ and show it is $90^\circ - \frac{1}{2}\angle C$, which is all you need. $\square$

5. Each of the positive integers $a_1, a_2, \ldots, a_n$ is less than 2016, and the least common multiple of any two is greater than 2016. Show that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 1 + \frac{n}{2016}.$$

Solution. By considering multiples of the a_i which are less than 2016 (these don't overlap by condition) we derive

$$\sum \left\lfloor \frac{2016}{a_i} \right\rfloor \leq 2016.$$

Upon using the fact that $\lfloor x \rfloor > x - 1$, we then obtain

$$\sum \left(\frac{2016}{a_i} - 1 \right) < 2016.$$

which rearranges to the desired conclusion. $\square$

6. Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers which satisfies

$$a_{n+1} \geq a_n^2 + \frac{1}{5}$$

for every positive integer n. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}$ for each positive integer n.

Solution. From the given we can deduce that

$$a_{n+1} \geq a_n^2 + \frac{1}{4} - \frac{1}{20} \geq a_n - \frac{1}{20}.$$

Thus for any n we have

$$a_{n+5} \geq a_{n+1} - 4 \cdot \frac{1}{20} = a_{n+1} - \frac{1}{5} = a_n^2.$$

Thus $\sqrt{a_{n+5}} \geq a_n \geq a_{n-5}$ follows. $\square$

7. Prove that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{m+1}{n} + \frac{n+1}{m}$$

is an integer.

Solution. In fact there are infinitely many pairs (m, n) for which

$$\frac{m+1}{n} + \frac{n+1}{m} = 3.$$

To see this, note that $(2, 3)$ is a solution which gives 3. Thereafter, we observe that the equation writes as

$$3mn = m^2 + n^2 + m + n \quad \text{or} \quad m^2 + (1 - 3n)m + (n^2 + n) = 0.$$

Thus by the so-called method of "Vieta jumping", if (m, n) is a solution with $m < n$, we obtain another solution $(3n - 1 - m, n)$ or $(n, 3n - 1 - m)$; this solution has greater sum than any previous one.

In this way, we generate an infinite chain of solutions:

$$(2, 3) \to (3, 6) \to (6, 14) \to (14, 35) \to \dots$$

□