Berkeley Math Circle: Monthly Contest 1 Solutions

1. Find the number of divisors of $2^9 \cdot 3^{14}$.

   Solution. Each divisor is the product of some power of 2 and some power of 3. There are 10 choices for the power of 2 ($2^0, 2^1, \ldots, 2^9$) and 15 choices for the power of 3 ($3^0, 3^1, \ldots, 3^{14}$), and, since these choices are independent, there are a total of $10 \cdot 15 = 150$ divisors. $\square$

2. Find all ordered triples (a, b, c) of positive integers with $a^2 + b^2 = 4c + 3$.

   Solution. There are no such ordered triples. Since the right side is odd, one of a and b is odd and the other even. But the square of any even number is a multiple of 4, and the square of any odd number has a remainder of 1 when divided by 4. But the right side leaves a remainder of 3 when divided by 4, a contradiction. Thus, this is impossible. $\square$

3. In the game *Kayles*, there is a line of bowling pins, and two players take turns knocking over one pin or two adjacent pins. The player who makes the last move (by knocking over the last pin) wins.

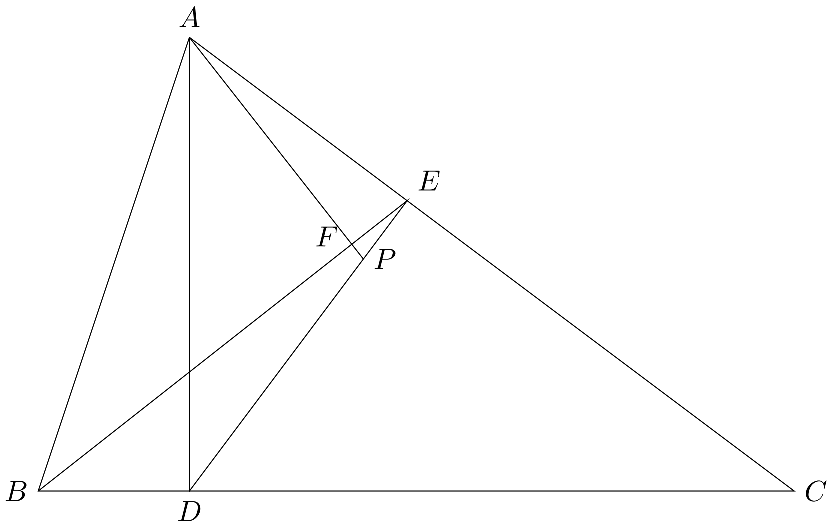
   Show that the first player can always win no matter what the second player does.

   (Two pins are *adjacent* if they are next to each other in the original lineup. Two pins do *not* become adjacent if the pins between them are knocked over.)

   Solution. On their first move, the first player knocks over the middle pin (if the number of pins is odd) or two pins (if the number is even). Then, they simply mirror what the second player does, and they will have a move as long as the second player does. $\square$

4. In $\triangle ABC$, points D and E lie on side BC and AC respectively such that $AD \perp BC$ and $DE \perp AC$. The circumcircle of $\triangle ABD$ meets segment BE at point F (other than B). Ray AF meets segment DE at point P. Prove that $DP/PE = CD/DB$.

   Solution. Since $\triangle DEC$ and $\triangle ADC$ are both right triangles and have common $\angle C$, we have
   $$\triangle DEC \sim \triangle ADC \implies \frac{DE}{CE} = \frac{AD}{CD}.$$
   Also, since $ABDF$ is cyclic, $\angle DBF = \angle DAF$ and $\angle AFB = \angle ADB = 90°$.



Thus, $\triangle AFE$ and $\triangle AEP$ are both right triangles with common $\angle FAE$, so

$$\angle AEF = \angle APE \implies \angle APD = CEB.$$

Now, by AA,

$$\triangle DPA = \triangle CEB \implies \frac{DP}{DA} = \frac{CE}{CB}.$$

Multiplying equalities gives

$$\frac{DP}{CD} = \frac{DP}{AD} \cdot \frac{AD}{CD} = \frac{CE}{CB} \cdot \frac{DE}{CE} = \frac{DE}{CB}$$

$$\implies \frac{DP}{DE} = \frac{CD}{BC} \implies \frac{DP}{PE} = \frac{CD}{DB}.$$ □

5. Show that for positive real numbers a, b, and c,

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)}.$$

Solution. Expanding and rearranging the denominator gives

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ac = c^2(a+b) + a^2(b+c) + b^2(c+a).$$

By Cauchy-Schwarz,

$$(c^2(a+b) + a^2(b+c) + b^2(c+a))\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq (a+b+c)^2,$$

and dividing by $c^2(a+b) + a^2(b+c) + b^2(c+a)$ gives the desired inequality. □

6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(f(x) + xf(y)) = x + f(x)y,$$

where $\mathbb{Q}$ is the set of rational numbers.

Solution. Setting $x = 0$ gives

$$f(f(0)) = f(0)y.$$

Since this must be true for all y, the only way this can happen is if $f(0) = 0$. Now, setting $y = 0$ gives

$$f(f(x)) = x$$

for all x. Setting $x = y = -1$ gives

$$f(f(-1) - f(-1)) = f(0) = 0 = -1 - f(-1) \implies f(-1) = -1.$$

Setting $x = -1$ gives

$$f(f(-1) - f(y)) = f(-1 - f(y)) = -1 - y.$$

Applying f to both sides and using $f(f(x)) = x$ gives

$$f(f(-1 - f(-y))) = -1 - f(-y) = -1 - y.$$

From here, we can use $f(-1) = -1$ and induction to show that $f(n) = n$ for all integers n. Now consider any rational number $y = \frac{m}{n}$. Setting $x = n$ gives

$$f\left(f(n) + nf\left(\frac{m}{n}\right)\right) = n + f(n) \cdot \frac{m}{n}$$

$$\implies n + nf\left(\frac{m}{n}\right) = f\left(n + n \cdot \frac{m}{n}\right) = n + m \implies f\left(\frac{m}{n}\right) = \frac{m}{n}.$$

Thus, we must have $f(x) = x$ for all $x \in \mathbb{Q}$. Clearly, this function works, so it is the unique solution. □

7. Evaluate the sum

$$\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\frac{P_i - 1}{P_{i+1}}\right) = \frac{1}{3} + \frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{7} + \frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{7}\cdot\frac{6}{11} + \dots,$$

where P_n denotes the n^{th} prime number.

Solution. Rewrite the given sum as

$$\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{5} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7} + \cdot\frac{1}{11} + \dots$$

$$= \frac{1}{3} + \left(1 - \frac{1}{3}\right)\frac{1}{5} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\frac{1}{7} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\frac{1}{11} + \dots$$

$$= \frac{1}{3} + \frac{1}{5} - \frac{1}{3\cdot 5} + \frac{1}{7} - \frac{1}{3\cdot 7} - \frac{1}{5\cdot 7} + \frac{1}{3\cdot 5\cdot 7} + \frac{1}{11} - \frac{1}{3\cdot 11}\frac{1}{5\cdot 11} - \frac{1}{7\cdot 11} + \dots$$

Note that each term in this sum must have denominator of the form $p_1p_2\dots p_k$, where the p_i are distinct primes (WLOG $p_1 < p_2 < \dots < p_k$). Further, each denominator occurs exactly once, namely, in expanding the product whose last term is $\frac{1}{p_k}$, and its sign is $(-1)^{k-1}$, since it comes from the term

$$\left(-\frac{1}{p_1}\right)\left(-\frac{1}{p_2}\right)\cdots\left(-\frac{1}{p_{k-1}}\right)\frac{1}{p_k}$$

in the expansion of the product containing $\frac{1}{p_k}$. However, this lets us rewrite the left side as

$$1-\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots.$$

However, we can see that

$$\frac{1}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)}=\frac{1}{1-\frac{1}{3}}\cdot\frac{1}{1-\frac{1}{5}}\cdot\frac{1}{1-\frac{1}{7}}\cdots$$

$$=\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots\right)\left(1+\frac{1}{7}+\frac{1}{7^2}+\ldots\right)=\sum_{n=1}^{\infty}\frac{1}{2n+1}.$$

Thus, since the Harmonic series diverges, the product

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots$$

must equal 0, so the desired sum is $1-0=1$. □