Berkeley Math Circle: Monthly Contest 2 Solutions

1. Carl computes the number

$$N = 5^{555} + 6^{666} + 7^{777}$$

and writes it in decimal notation. What is the last digit of N that Carl writes?

Solution. We look at the last digit of each term.

- The last digit of $5^\bullet$ is always 5.
- The last digit of $6^\bullet$ is always 6.
- The last digit of $7^\bullet$ cycles 7, 9, 3, 1, 7, 9, 3, ...

So the last digits are 5, 6, 7 in that order. Since $5 + 6 + 7 = 18$, the answer is 8. □

2. Given that

$$\begin{aligned} a + b &= 23 \\ b + c &= 25 \\ c + a &= 30 \end{aligned}$$

determine (with proof) the value of abc.

Solution. We add all three equations to obtain

$$2(a + b + c) = 78$$

so $a + b + c = 39$. Therefore,

$$\begin{aligned} a &= 39 - 25 = 14 \\ b &= 39 - 30 = 9 \\ c &= 39 - 23 = 16. \end{aligned}$$

Thus $abc = 14 \cdot 9 \cdot 16 = 2016$. □

3. In a standard 52 deck of cards, there are 13 cards of each of four suits. Kevin guesses the suit of the top card, and the top card is revealed and discarded. This process continues till there are no cards remaining.

If Kevin always guesses the suit of which there are the most remaining (breaking ties arbitrarily), prove that he will get at least 13 guesses right.

Solution. Imagine that the cards have been given ranks 1, 2, ..., 13 and moreover that within each rank the cards have been sorted in ascending order (i.e. Kevin will encounter 1, 2, ..., 13 of hearts in that order).

Then, observe that Kevin will always guess the last card of rank r correctly, for any $r = 1, \ldots, 13$. This completes the proof. □

4. Find all triples of continuous functions f, g, h from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x+y) = g(x) + h(y)$ for all real numbers x and y.

Solution. The answer is $f(x) = cx + a + b$, $g(x) = cx + a$, $h(x) = cx + b$, where a, b, c are real numbers. Obviously these solutions work, so we wish to show they are the only ones.

First, put $y = 0$ to get $f(x+0) = g(x) + h(0)$, so $g(x) = f(x) - h(0)$. Similarly, $h(y) = f(y) - g(0)$. Therefore, the functional equation boils down to $f(x+y) = f(x)+f(y)-(g(0)+h(0))$. By shifting and appealing to Cauchy's functional equation (with f continuous) we get $f(x) = cx + g(0) + h(0)$, $g(x) = cx + g(0)$ and $h(x) = cx + h(0)$. $\square$

5. Let x, y, z be positive numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Show that

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Solution. It's equivalent to show

$$\sum_{\text{cyc}} \sqrt{x + yz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)} \geq \sum_{\text{cyc}} \sqrt{x} + \sqrt{xyz} \cdot \frac{1}{x}$$

but now we're done by Cauchy as

$$\sqrt{x + yz\left(\frac{1}{x} + \frac{1}{z} + \frac{1}{z}\right)} = \frac{\sqrt{(x+y)(x+z)}}{\sqrt{x}} \geq \frac{x + \sqrt{yz}}{\sqrt{x}} = \sqrt{x} + \frac{\sqrt{xyz}}{x}.$$

$\square$

6. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle BCD = \angle DEA = 90°$. Point F lies on AB such that $\frac{AF}{AE} = \frac{BF}{BC}$. Prove that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.

Solution. Let ω denote the circumcircle of $\triangle CDE$ and let D_1 be the point opposite to D. Let DA meet ω at A_1 and let $F' = A_1C \cap AB$. If we let $\alpha = \angle DA_1C = \angle DD_1C = \angle ED_1D$ then

$$\frac{AF'}{AE} = \frac{AF' \cdot AD_1}{AA_1 \cdot AD} = \frac{\sin\alpha \cdot \sin\angle D_1DA}{\sin\angle A_1F'A \cdot \sin\alpha} = \frac{\sin\angle D_1CA}{\sin\angle CF'B} = \frac{F'B}{BC}.$$

Hence $F = F'$. Now $\angle F'CE = \angle A_1CE = \angle A_1DE = \angle ADE$. $\square$

7. Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)^2 - f(y)^2 = f(x+y)f(x-y)$$

for all real numbers x and y.

Solution. The answer is $f(x) = kx$, $f(x) = a\sin(cy)$, $f(x) = a\sinh(cy)$, where $a, c \in \mathbb{R}$. The given functional equation is

$$f(x)^2 - f(y)^2 = f(x+y)f(x-y).$$

Observe that $x = y = 0$ gives $f(0) = 0$.

Since f is smooth, we may differentiate with respect to x and obtain

$$2f'(x)f(x) = f'(x+y)f(x-y) + f(x+y)f'(x-y).$$

Now differentiate this with respect to y to obtain

$$\begin{aligned} 0 &= \left[f''(x+y)f(x-y) - f'(x+y)f'(x-y)\right] \\ &\quad + \left[f'(x+y)f'(x-y) - f(x+y)f''(x-y)\right] \\ &= f''(x+y)f(x-y) - f(x+y)f''(x-y) \end{aligned}$$

From this we conclude the key relation: for any real numbers X and Y:

$$f''(X)f(Y) = f(X)f''(Y).$$

Assume f isn't identically zero. Then we deduce there's a constant k such that

$$f''(x) = kf(x)$$

for all x.

This is a standard differential equation with cases on k.

- If $k = 0$, the solution set is $f(x) = ax + b$. Then $f(0) = 0 \implies b = 0$, and we can check $f(x) = ax$ works.
- If $k < 0$, the solution set is $f(x) = a\sin(-kx) + b\cos(-kx)$. Again $f(0) = 0 \implies b = 0$, and we can check $f(x) = a\sin(-kx)$ works.
- If $k > 0$, the solution set is $f(x) = a\sinh(-kx) + b\cosh(-kx)$. Again $f(0) = 0 \implies b = 0$, and we can check $f(x) = a\sinh(kx)$ works.

□