Berkeley Math Circle: Monthly Contest 4
Due January 22, 2020

Instructions

• This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Contest 4 is due on January 22, 2020.

• Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

   BMC Monthly Contest 4, Problem 2
   Evan O’Dorney
   Grade 3, BMC Beginner
   from Springfield Middle School, Springfield

Submit different problems on different pages as they are graded separately.

• Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

• Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at http://mathcircle.berkeley.edu for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 4

1. Prove that $\sqrt{n+1} + \sqrt{n}$ is irrational for every positive integer $n$.

2. Suppose a sequence $s_1, s_2, \ldots$, of positive integers satisfies $s_{n+2} = s_{n+1} + s_n$ for all positive integers $n$ (but not necessarily $s_1 = s_2 = 1$). Prove that there exists an integer $r$ such that $s_n - r$ is not divisible by 8 for any integer $n$.

3. For positive real numbers $a, b, c$ satisfying $ab + bc + ca = 1$, prove that

   \[ \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a^2 + b^2 + c^2 + 2. \]

4. Suppose we have a convex polygon in which all interior angles are integers when measured in degrees, and the interior angles at every two consecutive vertices differ by exactly $1^\circ$. If the greatest and least interior angles in the polygon are $M^\circ$ and $m^\circ$, what is the maximum possible value of $M - m$?
5. Given a quadrilateral $ABCD$ extend $AD$ and $BC$ to meet at $E$ and $AB$ and $DC$ to meet at $F$. Draw the circumcircles of triangle $ABE$, $ADF$, $DCE$, and $BCF$. Prove that all four of these circles pass through a single point.

6. Determine, with proof, whether or not there exist distinct positive integers $a_1$, $a_2$, \ldots, $a_n$ such that

\[
\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 2019.
\]

7. A simple graph $G$ on 2020 vertices has its edges colored red and green. It turns out that any monochromatic cycle has even length. Given this information, what is the maximum number of edges $G$ could have?