Berkeley Math Circle: Monthly Contest 4

Due January 23, 2018

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 4 is due on January 23, 2018.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 4

1. When the number
$$N = 1^1 \times 2^2 \times 3^3 \times \cdots \times 9^9$$
is written as a decimal number, how many zeros does it end in?

2. A square and an equilateral triangle have the property that the area of each is the perimeter of the other. What is the area of the square?

3. Find all the ways which one can assign an integer to each vertex of a 100-gon subject to the following condition: among any three consecutive numbers written down, one of the numbers is the sum of the other two.

4. Give an example of a *strictly increasing* function $f : \mathbb{R} \to [0, 1]$ with the property that
$$f(x + y) \leq f(x) + f(y)$$
for any real numbers x and y.

5. Louis moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -4y)$, $(-2x, 5y)$, $(x + 1, y + 6)$ and $(x - 7, y)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

6. A sequence a_1, a_2, ... of positive integers satisfies $a_1 = 1$ and

$$a_{n+1} = 2^{a_n} + a_n$$

for $n \geq 1$. Prove that a_1, a_2, ..., a_{243} leave distinct remainders when divided by 243.

7. Let ABC be a triangle with incenter I and circumcenter O for which $BC < AB < AC$. Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Prove that $\overline{DE} \perp \overline{IO}$.