Berkeley Math Circle: Monthly Contest 8

Due May 2, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 8 is due on May 2, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 8, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 8

1. Give an example of four positive integers a, b, c, d, no two of which are the same, satisfying $a^2 + b^2 = c^2 + d^2$.

2. Let $n > 1$ be a positive integer. Show that $n^4 + n^2 + 1$ is not a prime number.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$3f(x + 2y + 3z) \leq f(x + y) + f(y + z) + f(z + x)$$
for all real numbers x, y, z.

4. Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Show that there exists a point P such that $\angle PAX = \angle PBY = \angle PCZ = 90°$.

5. Let ABC be an acute triangle and let ℓ be a line in the plane of triangle ABC. We've drawn the reflection of the line ℓ over the sides AB, BC and AC and they intersect in the points A', B' and C'. Prove that the incenter of the triangle $A'B'C'$ lies on the circumcircle of the triangle ABC.

6. Two sequences of integers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ satisfy

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for all $n = 3, 4, \ldots$. Prove that there exists an integer k such that $a_k = a_{k+1000}$.

7. Find all positive integers n such that when the nth harmonic number

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

is written in lowest terms, the numerator of H_n is divisible by 3.