Berkeley Math Circle: Monthly Contest 6

Due March 14, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 14, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 6

1. Prove that
$$n(n+1)(2n+1)$$
is always divisible by 6, for n a positive integer.

2. Oscar draws a triangle ABC on a sheet of paper. He finds that the side lengths of ABC are all powers of 2 (i.e. among 1, 2, 4, 8, . . .). Prove that Oscar's triangle is isosceles.

3. Let a, b, c, d be positive integers such that $ab = cd$. Prove that $a + b + c + d$ is not a prime number.

4. Prove that there exists an infinite sequence of a_1, a_2, . . . positive integers such that the following condition holds: $\gcd(a_m, a_n) = 1$ if and only if $|m - n| = 1$.

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

6. A bulldozer is touring Pascal's triangle. It starts at the top of the triangle, at $\binom{0}{0} = 1$. Each move, it travels to an adjacent positive integer, but can never return to a spot it has already visited. Moreover, if it has visited two numbers $a > b$, it may not visit $a + b$ or $a - b$. Finally, the bulldozer is confined to the first 140 rows of Pascal's triangle.

   Prove that the bulldozer may visit at least 2017 numbers. (By convention, the nth row contains the entries $\binom{n-1}{k}$ for $k = 0, \ldots, n-1$, hence the nth row has n entries.)

7. We wish to place ways exactly 100 dominoes (of size 2×1 or 1×2) without overlapping on a 20×20 chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column. In how many ways can this be done?