Berkeley Math Circle: Monthly Contest 3

Due December 5, 2017

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 3 is due on December 5, 2017.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 3

1. Find all ordered pairs (x, y) of positive integers such that $xy - x - y = 11$.
2. How many numbers are there from 1 to 100 that are neither a multiple of 7 nor contain the digit 7?
3. Show that for any five points in the plane, no three of which are collinear, some four form a convex quadrilateral.
4. In the game *Sprouts*, there are initially n spots drawn on a plane, and on each move two spots are connected with an edge and a new spot is drawn on this edge. No two edges can cross, and no spot may have more than 3 edges coming from it. An edge may be drawn from a spot to itself. The game ends when no more moves can be made.
   a) Show that the game must end in at most $3n - 1$ moves.
   b) Show that the game will last at least $2n$ moves.

5. Let $ABCDEFG$ be a regular heptagon. Let X be the intersection of diagonals BE and CG, and let Y be the intersection of BD and CE. Prove that A, X, and Y are collinear.

6. Let $\phi(n)$ be the number of positive integers less than or equal to n and relatively prime to n. Evaluate
$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n}.$$

7. Find all pairs of prime numbers (p, q) such that $p^2 - p - 1 = q^3$.