Berkeley Math Circle: Monthly Contest 2

Due November 8, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on November 8, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 2

1. Carl computes the number

   $$N = 5^{555} + 6^{666} + 7^{777}$$

   and writes it in decimal notation. What is the last digit of N that Carl writes?

2. Given that

   $$a + b = 23$$
   $$b + c = 25$$
   $$c + a = 30$$

   determine (with proof) the value of abc.

3. In a standard 52 deck of cards, there are 13 cards of each of four suits. Kevin guesses the suit of the top card, and the top card is revealed and discarded. This process continues till there are no cards remaining.

   If Kevin always guesses the suit of which there are the most remaining (breaking ties arbitrarily), prove that he will get at least 13 guesses right.

4. Find all triples of continuous functions f, g, h from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x+y) = g(x) + h(y)$ for all real numbers x and y.

5. Let x, y, z be positive numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Show that

$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

6. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle BCD = \angle DEA = 90°$. Point F lies on AB such that $\frac{AF}{AE} = \frac{BF}{BC}$. Prove that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.

7. Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)^2 - f(y)^2 = f(x+y)f(x-y)$$

for all real numbers x and y.