Berkeley Math Circle: Monthly Contest 1

Due October 4, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on October 4, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 1, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 1

1. Rachel measures the angles of a certain pentagon $ABCDE$ in degrees. She finds that $\angle A < \angle B < \angle C < \angle D < \angle E$, and also that the angle measures form an *arithmetic progression*, meaning that $\angle B - \angle A = \angle C - \angle B = \angle D - \angle C = \angle E - \angle D$.

   What was the measure of $\angle C$?

2. Victor has four red socks, two blue socks, and two green socks in a drawer. He randomly picks two of the socks from the drawer, and is happy to see that they are a matching pair. What is the probability the pair was red?

3. Kelvin the frog jumps along the number line starting at 0. Every time he jumps, he jumps either one unit left or one unit right. For example, one sequence of jumps might be $0 \to -1 \to 0 \to 1 \to 2 \to 3 \to 2$.

   How many ways are there for Kelvin to make exactly 10 jumps and land on a prime number? (The prime numbers are the sequence $2, 3, 5, 7, \ldots$. Negative numbers are not considered prime.)

4. Decide whether that there exists an infinite set S of positive integers with the property that if we take any finite subset T of S, the sum of the elements of T is not a perfect kth power for any $k \geq 2$.

5. Solve for real x:

$$x + \sqrt{(x+1)(x+2)} + \sqrt{(x+2)(x+3)} + \sqrt{(x+3)(x+1)} = 4.$$

6. On a circle we write $2n$ real numbers with a positive sum. For each number, there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both these sets.

7. Let $ABCD$ be a convex quadrilateral. Assume that the incircle of triangle ABD is tangent to $\overline{AB}$, $\overline{AD}$, $\overline{BD}$ at points W, Z, K. Also assume that the incircle of triangle CBD is tangent to $\overline{CB}$, $\overline{CD}$, $\overline{BD}$ at points X, Y, K. Prove that quadrilateral $WXYZ$ is cyclic.