24th Bay Area Mathematical Olympiad

BAMO-8 Exam

March 1, 2023

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

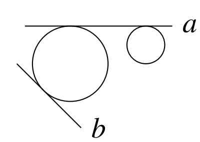
Please label all pages that you submit for grading in the following way: each new problem should start with a title page (which you should have received). Write your identification number in the box on the left and write your identification number *again* in the box on the right, but with the letter "A" at the beginning. For example, if your id # is 31416, then you write **31416** in the left box and **A31416** in the right box. (Your papers will be scanned and these duplicate ID #s will help the grading platform recognize your ID #.) If your work uses extra pages, use plain blank white paper and write the problem letter, your ID# and what page it is on the **bottom** of the page. For example, write "Problem A, p. 2 of 3, ID 31416." Your supervisor may have provided you with "continuation" pages which are already formatted for you to provide this information.

Write neatly. If your paper cannot be read, it cannot be graded! *Please write only on one side of each sheet of paper.*

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

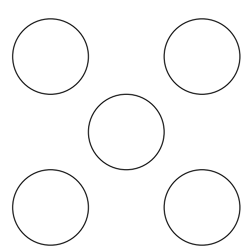
Problems A and B are on this side; problems C, D, and E are on the other side.

A A *tangent line* to a circle is a line that intersects the circle in exactly one point. A *common tangent line* to two circles is a line that is tangent to both circles. As an example, in the figure to the right, line a is a common tangent to both circles, but line b is only tangent to the larger circle.



Given two distinct circles in the plane, let n be the number of common tangent lines that can be drawn to these two circles. What are all possible values of n? Your answer should include drawings with explanations.

B Ara and Bea play a game where they take turns putting numbers from 1 to 5 into the cells of the X-shaped diagram on the right. Each number can be played only once, and a cell cannot have more than one number placed in it. Ara's goal is for the two diagonals of the X diagram to have the same sum when the game is over; Bea's goal is for these two sums to be unequal.



(a) Show that Ara can always win if he goes first.

(b) Show that Bea can always win if she goes first.

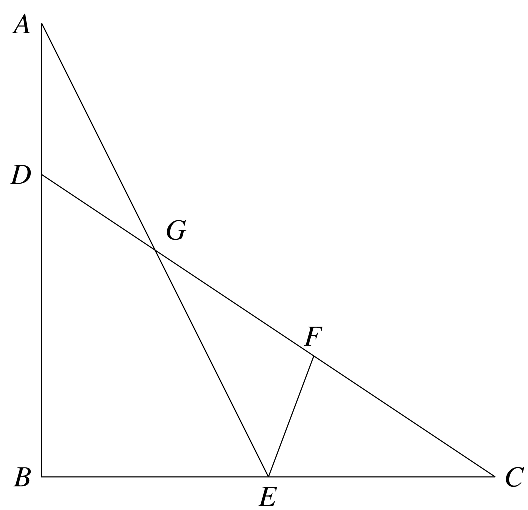
C Mr. Murgatroyd decides to throw his class a pizza party, but he's going to make them hunt for it first. He chooses eleven locations in the school, which we'll call $1,2,\ldots,11$. His plan is to tell students to start at location 1, and at each location n from 1 to 10, they will find a message directing them to go to location $n+1$; at location 11, there's pizza!

Mr. Murgatroyd sends his teaching assistant to post the ten messages in locations 1 to 10. Unfortunately, the assistant jumbles up the message cards at random before posting them. If the students begin at location 1 as planned and follow the directions at each location, show that they will still get to the pizza.

D Given a positive integer N (written in base 10), define its *integer substrings* to be integers that are equal to strings of one or more consecutive digits from N, including N itself. For example, the integer substrings of 3208 are 3, 2, 0, 8, 32, 20, 320, 208, and 3208. (The substring 08 is omitted from this list because it is the same integer as the substring 8, which is already listed.)

What is the greatest integer N such that no integer substring of N is a multiple of 9? (Note: 0 is a multiple of 9.)

E In the following figure—not drawn to scale!—E is the midpoint of BC, triangle FEC has area 7, and quadrilateral $DBEG$ has area 27. Triangles ADG and GEF have the same area, x. Find x.



You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2023 Awards Ceremony**, which will be held at Santa Clara University in the afternoon on Sunday, March 19. This event will include a mathematical talk by **Inna Zakharevich (Cornell University)**, and the awarding of prizes. Please check with your proctor and/or `https://www.bamo.org` for a more detailed schedule, plus directions.