12th Bay Area Mathematical Olympiad

BAMO-8 Exam

February 23, 2010

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The four problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

A We write $\{a,b,c\}$ for the set of three different positive integers a, b, and c. By choosing some or all of the numbers a, b and c, we can form seven nonempty subsets of $\{a,b,c\}$. We can then calculate the sum of the elements of each subset. For example, for the set $\{4,7,42\}$ we will find sums of 4, 7, 42, 11, 46, 49, and 53 for its seven subsets. Since 7, 11, and 53 are prime, the set $\{4,7,42\}$ has exactly three subsets whose sums are prime. (Recall that prime numbers are numbers with exactly two different factors, 1 and themselves. In particular, the number 1 is *not* prime.)

What is the largest possible number of subsets with prime sums that a set of three different positive integers can have? Give an example of a set $\{a,b,c\}$ that has that number of subsets with prime sums, and explain why no other three-element set could have more.

B A *clue* "k digits, sum is n" gives a number k and the sum of k distinct, nonzero digits. An *answer* for that clue consists of k digits with sum n. For example, the clue "Three digits, sum is 23" has only one answer: 6,8,9. The clue "Three digits, sum is 8" has two answers: 1,3,4 and 1,2,5.

If the clue "Four digits, sum is n" has the largest number of answers for any four-digit clue, then what is the value of n? How many answers does this clue have? Explain why no other four-digit clue can have more answers.

Please turn over for the remaining problems!

C Suppose a,b,c are real numbers such that $a+b \geq 0$, $b+c \geq 0$, and $c+a \geq 0$. Prove that

$$a+b+c \geq \frac{|a|+|b|+|c|}{3}.$$

(Note: $|x|$ is called the *absolute value* of x and is defined as follows. If $x \geq 0$ then $|x| = x$; and if $x < 0$ then $|x| = -x$. For example, $|6| = 6$, $|0| = 0$ and $|-6| = 6$.)

D Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red.

Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2010 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 7. This event will include lunch, a mathematical talk by Thomas Banchoff of Brown University, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.