11th Bay Area Mathematical Olympiad

BAMO-8 Exam

February 24, 2009

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

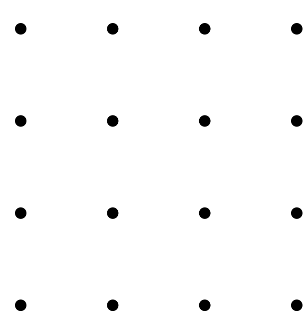
Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The four problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 A square grid of 16 dots (see the figure) contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?

Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.



2 The *Fibonacci sequence* is the list of numbers that begins 1, 2, 3, 5, 8, 13 and continues with each subsequent number being the sum of the previous two.

Prove that when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example,

$$1-2+3-5=-3,$$

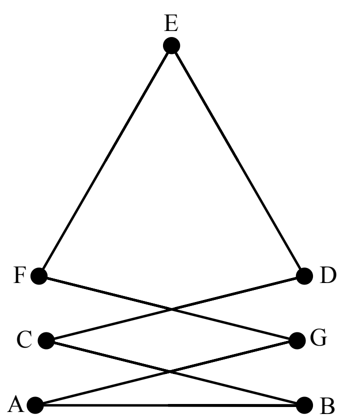
and 3 is an element of the Fibonacci sequence.

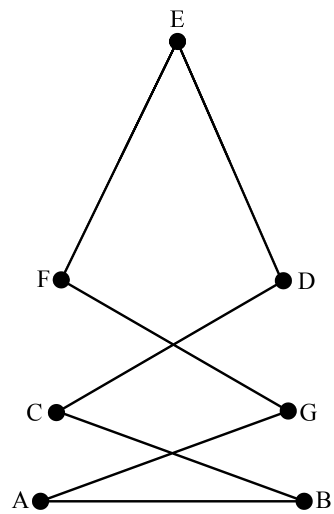
3 There are many sets of two different positive integers a and b, both less than 50, such that a^2 and b^2 end in the same last two digits. For example, $35^2 = 1225$ and $45^2 = 2025$ both end in 25. What are all possible values for the average of a and b?

Note that for the purposes of this problem, single-digit squares are considered to have a leading zero, so for example we consider 2^2 to end with the digits 04, not 4.

4 Seven congruent line segments are connected together at their endpoints as shown in the figure below at the left. By raising point E the linkage can be made taller, as shown in the figure below and to the right. Continuing to raise E in this manner, it is possible to use the linkage to make A, C, F, and E collinear, while simultaneously making B, G, D, and E collinear, thereby constructing a new triangle ABE.

Prove that a regular polygon with center E can be formed from a number of copies of this new triangle ABE, joined together at point E, and without overlapping interiors. Also find the number of sides of this polygon and justify your answer.





You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2009 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 8. This event will include lunch, a mathematical talk by Francis Su of Harvey Mudd, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.