Fuzzy Calculus
From Probability to Geometry

I. Warm-up/Review

1. Fill in the table at home

n	$b_n = \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + ... + \frac{1}{n!}$	$c_n = \sum_{k=0}^{n} \frac{2^k}{k!} = \frac{2^0}{0!} + \frac{2^1}{1!} + ... + \frac{2^n}{n!}$
1		
2		
4		
6		
8		

2. Probability: what is our average earning for the following dice game?
   a) Throw a fair dice and get 10 dollars if 5 or 6, lose 2 dollars otherwise?

   b) Throw two fair dice and get 10 dollars if 9 or higher, lose 2 dollars otherwise?

c) What is the overall formula to calculate the probability in this type of problems where there is a certain value attached to each outcome?

3. In a card game, Harry wins \$2 if they draw a 10 or higher and lose \$1 otherwise. Every time a card is drawn, it is removed from the deck. The game ends when Harry first draws a losing card. On average, how much is Harry's earnings/loss per game?

4. What is the probability that when you randomly pick 2 real numbers between -1 and 1, the sum of their squares is at most 1?

II. Expansion from Previous Times

1. Distance - Circle
   a) Can we interpret the locus definition of the *unit circle* in an algebraic way? This result would be called the "equation" of the circle.
   b) Can we find an algebraic way to express the unit disc (all points on and inside the circle)?
2. Describe a sphere algebraically? How about the unit ball in 3-dimension?

III. The Dart Game

Harry is training to join a dart league. So they sets up a game with two motivations:

- The game should get harder as it goes.
- The closer they hit to the bullseye, the better the rewards.

So let's agree on the following game:

1. To encourage Harry to hit the bullseye, at the first throw, we are going to make the entire dartboard the bullseye.
2. For the next throw, we will shrink the bullseye by making the new radius half the perpendicular chord through the previous hit, leaving the square the same size.
3. We continue shrinking the bullseye after every hit. For **every throw** (not every hit), Harry gets a dollar from each student in the class. The game ends when Harry misses.

> The major question that we will answer: On average, how much money does Harry make per game?

IV. Balls of high-dimension

Look up the formula for volume of a 4-dimensional, 5-dimensional,... ball and fill in the table below:

n	Volume of the n-dimensional ball with radius 1
3	
4	
5	
6	
8	

DAY 3 - INFINITY AND IN-BETWEEN PROBABILITIES

V. How large is ∞?

Iffy - Def
We use infinity to denote ....

Some operations we can work with infinity are as below:

Is infinity unique? We will attempt to use the concept of probability to examine this new "creature"!

VI. Discrete Probs

1. What is the probability that if we randomly pick a digit that we pick something divisible by 3? What if we pick from the set $\{1, 2, 3, ..., 14, 15\}$?

2. (Plicker) What is the probability that when we randomly pick a natural number we end up picking 0?

3. (Plicker) What is the probability that if we randomly pick a natural number then we pick something divisible by 3?

4. (Plicker) What is the probability that if we randomly pick an integer then we pick something natural?

5. (Plicker) What is the probability that if we randomly pick a rational number then we pick something natural?

6. (Plicker) What is the probability that if we randomly pick a real number then we pick something natural?

7. (Plicker) What is the probability that if we randomly pick a real number then we pick something irrational?

VII. How many numbers are there?

1. How many natural numbers are there?
2. How many integers are there?
3. How many rational numbers are there?
4. Are there the same amount of numbers in all three parts above?

5. How many real numbers are there? Is this the same kind of ∞? Why or Why not?

VIII. Analysis on numbers, points and dimensions

By now we should see that in order to convincingly calculate a probability, we have to establish a standard of measure for whichever set we are working with. Formally, this will tie into *Measure Theory*, which lays the very foundation for *Probability Theory*. We are not going to develop this entire theory at the moment, but we will attempt to bring some motivation, inspiration and consistency towards the theory.

1. In the universe of rational numbers, we can work with just counting the quantity of numbers, as the largest possible quantity (fancy math word = *cardinality*) is *countable*, $\aleph_0$.

2. When we work with the real numbers, we now introduce another "dimension," as **every real number is uniquely a point on a line**, ie the set all of real numbers, with its ordering, forms a line (that's why we call it the real-line!). This 1-dimensional set of numbers will take "length" as the standard of measure.
   a) Each point, say 5, 6, 7, has length ......
   b) The set of natural numbers has length ......
   c) The segment (interval) $[0, 1]$ has length ....
   d) Then, the question of probability when randomly picking a real number becomes a question about what "length" your desired selection has?

3. If we work in the plane, the standard of measure for the 2D objects is .... Therefore, the typical question of probability of picking a point in the plane (or any region in the plane), becomes a question about .... of the desired selection.

a) What is the probability of hitting a disc inside a square?

b) What is the probability of landing on the circle inscribed in the square?

c) What is the probability of hitting the open disc (no boundary) inscribed in the square?

d) What is the probability of hitting the bullseye in a "random-throw" dart game?

IX. Set-up to Solve the Dart Game

1. Following the idea of Expected Value, what would be the set up to find Harry's Average Earning per game?

2. Doesn't that seem too hard to find each $P(T = k)$ for each k? How about we try something slightly easier? What is $P(T \geq 2)$?

3. Following question 2 above, write down an algebraic condition for $P(T \geq 3)$. You may not know how find it yet.

4. Generalizing to $P(T \geq k)$.

X. Balls of high-dimension

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1. The **Gamma Function**: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. People say Gamma Function is the extension of the Factorial Function. Why?

2. The **Beta Function**: $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. Are Gamma and Beta related?

3. Free fact (or you can prove it yourself): $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

4. What is the volume of the $2n$-dimensional unit ball?

5. (Take-home) What is its surface area of the $2n-$dimensional unit sphere?