Strolling with Euler: Extradimensional Exploration

Let's start with a simple question. How do we find the sum of the interior angles of a convex polygon?

- List as many methods as you can.
- Think about: how can you adapt your method to concave polygons?

Formula for the interior angles of a polygon: _______________.

Part 2 – Platonic Solids Again

Definition: A platonic solid is a polyhedron which a. has only one kind of regular polygon for a face and b. has the same number of faces at each vertex.

One simple example is that of a cube. Each face is a square (=regular quadrilateral) and each vertex is connected to exactly three squares.

In Schläfli notation, this is \{4, 3\}: the regular polyhedron with 3 regular 4-gons at each vertex.

- Can there be polyhedra with exactly one or two squares at each vertex?
- Can there be polyhedra with exactly four squares at each vertex?

For future reference, we'll need the notion of the “angular defect” as well. Rather than focusing on what is present at the vertex, we focus on what is absent: the deviation from 360 degrees at the vertex, or “leftover” angle. In this case, it is \(360 - 90 \times 3 = 90\).
Let's prove that there are exactly 5 Platonic solids and no more.

- Be careful – we'll need some way to handle discussing regular n-gons as n becomes large.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Schlāfli #</th>
<th>Defect at vertex</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
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<tr>
<td>Cube</td>
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<td>Octahedron</td>
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<td>Dodecahedron</td>
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<td>Icosahedron</td>
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Q: Can you find the combinatorial properties of a regular polytope from the Schlāfli number alone?
Part 3 – Duals, Truncations, Stellations, $\chi_e$

Let's try placing a point in the middle of each face and connecting the points. Maybe something interesting will happen.

Write a sentence explaining the pattern that you found:

Q: What happens to the Schläfli number of a polyhedron when you take the dual?

Two common operations on polyhedra, in addition to taking the dual, are stellation (=replacing each face with a pyramid) and truncation (=replacing each vertex with an appropriate regular polygon). Let's experiment with the snub and stellated cubes:

- Same as before: let's find the vertices, edges, faces, and angular defect at each vertex.
Try something more complicated: the truncated icosahedron, also known as a “soccer ball.”

- Control question – which polygon replaces each vertex?

We’ve produced some data. Let’s put it in a convenient table. Do you know the Euler Characteristic? While we’re at it, we might as well throw that in too.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Defect at vertex</th>
<th>V</th>
<th>Total Defect at all V</th>
<th>$\chi = V - E + F$</th>
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</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
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<td>Cube</td>
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<td>Icosahedron</td>
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<tr>
<td>Snub Cube</td>
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<td>Stellated Cube</td>
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<tr>
<td>Soccer Ball</td>
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</table>

Do you see a pattern? Describe it:
Part 4 – Euler Characteristic and Gauss-Bonnet Theorem

We may have a nice pattern over the figures that we've put in the table above, but it has only 8 entries! It would be great to have something a little bit more convincing.

Let's try to find the total angular defect and Euler Characteristic of, say, a donut shape built from cubes.

- Is this the same as finding the same values for a torus? Why or why not?

Total Angular Defect:

Euler Characteristic:

Our hypothesis:
Proof:

- Hint: Parts 1 and 2 will be useful here.

**Q (topology):** Can you find the Euler Characteristic of a Möbius strip? A Klein bottle?

The angular defect is not merely a nice tool for finding the certain features of polyhedra: it describes curvature and helps to explain how map distortions are created.

Topologically speaking, we will find that all polyhedra with equal euler characteristic are equivalent.

Further reading:

- Alexander Givental, “Geometry of Surfaces and the Gauss-Bonnet Theorem:”
  https://math.berkeley.edu/~giventh/difgem.pdf
- Coxeter, *Regular Polytopes*.
- Ueno, Shiga, Morita, *A Mathematical Gift, I: The interplay between topology, functions, geometry, and algebra*