Instructions: This text is for reading after the class. During class, pay attention. If you get bored, try some of the harder problems in the mixed problems section, or have a go at the unsolved problem in Section 4.

We consider sequences of numbers, e.g.

\[ 1, 2, 4, 8, 16, \ldots \]
\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]
\[ 1, 1, 2, 3, 5, 8, 13, \ldots \]

Generally we write \( a_0, a_1, a_2, \ldots \) or \( (a_n) \) for a sequence\(^2\).

For the first two sequences there are closed formulas: \( a_n = 2^n \) and \( a_n = \frac{1}{n+1} \) for all \( n \), respectively.

The third sequence is built from the law \( a_n = a_{n-1} + a_{n-2} \) (for \( n \geq 2 \)).

This is a recurrence relation, i.e. a rule by which \( a_n \) can be found from

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2 One also writes \( (a_n)_{n \in \mathbb{N}_0} \). In this text \( n \) will always start at \( n = 0 \).
previous sequence elements. We also speak of a recursively defined sequence. For a recurrence relation to determine a sequence uniquely you must specify initial values, in this case \( a_0 = 1, a_1 = 1 \).

1 Recursively defined sequences

There are many interesting problems connected with recursively defined sequences. For example:

1. Can we find a closed formula?

2. How do the \( a_n \) behave for large \( n \)? Do they approach some number, or infinity, as \( n \to \infty \)? (We then also say that this is the limit, and that \( a_n \) tends to this limit, and write \( a_n \to x \) if the limit is \( x \). If yes, how quickly? If they tend to infinity, then how fast (for example exponentially, polynomially)?

Problem 1. Find a closed formula for the sequences defined by:
\[
\begin{align*}
    a_n &= a_{n-1} + 2, \quad a_0 = 1; \\
    b_n &= 3b_{n-1}, \quad b_0 = 1; \\
    c_n &= 2c_{n-1} + 1, \quad c_0 = 2.
\end{align*}
\]

Often it is difficult or impossible to find a closed formula. Sometimes you can still find the behavior as \( n \to \infty \):

Example (Finding a limit from a recurrence relation).
\[
a_{n+1} = \frac{1}{2} \left( a_n + \frac{1}{a_n} \right), \quad a_0 = 2
\]

The first terms are \( 2, \frac{5}{4}, \frac{41}{30} \). This leads us to conjecture:

1. All \( a_n \) are bigger than \( 1 \).

2. The sequence is decreasing, i.e. \( a_{n+1} \leq a_n \) for all \( n \).

3. The terms of the sequence approach the limit \( 1 \).

The first two claims are easy to check. (In 1. use \( x + \frac{1}{x} \geq 2 \) for all \( x > 0 \), with equality if and only if \( x = 1 \).)

Then the third claim follows from the first two. This works in two steps.

\[ \text{We will not treat limits formally (that’s done in an analysis course). Here are some simple examples for intuition: } a_n = \frac{1}{n}, \text{ that is } 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \text{ tends to } 0, \text{ } a_n = n \text{ tends to } \infty \text{ and the sequence } 0, 1, 0, 1, 0, 1, \ldots \text{ has no limit.} \]
First, the following theorem holds. It should be intuitively clear (and is proved rigorously in any analysis class):

**Theorem:** A decreasing sequence which is bounded from below has a limit. (Of course there is a similar fact for increasing and bounded above.)

In the second step, we find the limit: Call the limit $x$. How do we calculate it? Since $x$ is the limit of the $a_n$, it is also the limit of the $a_{n+1}$, so we get from the recurrence relation for $n \to \infty$:

$$a_{n+1} = \frac{1}{2} \left( a_n + \frac{1}{a_n} \right)$$

↓

$$x = \frac{1}{2} \left( x + \frac{1}{x} \right)$$

This is the **fixed point equation**. A short calculation then gives $x = 1$ (or $x = -1$, but $x$ must be positive).

Therefore, the limit of the $a_n$ is 1.

**Problem 2.** Investigate how the sequence defined by $a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right)$, $a_0 = 5$ behaves as $n \to \infty$.

Use a calculator to find the first 6 terms of this sequence. You will observe that it approaches 1.4142... very quickly, which is $\sqrt{2}$. Why?

**Problem 3.** Investigate how fast the sequence in Problem 2 approaches its limit.

**Problem 4.** Investigate the behavior of the sequence defined by $a_{n+1} = 1 - a_n$, $a_0 = 0$ as $n \to \infty$. What is the solution of the fixed point equation?

We see: it is not sufficient to just solve the fixed point equation.

**Linear recurrence relations**

We want to find a closed formula for the **Fibonacci** numbers, which are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = F_1 = 1$$

The first few are 1, 1, 2, 3, 5, 8, ... .

Try to guess a formula! It seems impossible.

There are (at least) two systematic methods to find a closed formula:
1. By a power ansatz

2. Using generating functions.

The power ansatz is simpler but generating functions are more powerful: they can be used for more problems, as we will see!

**Solving the Fibonacci recurrence relation by a power ansatz**

**Step 1:** At first, forget about the initial condition. Just try to find a solution of the recurrence relation $a_n = a_{n-1} + a_{n-2}$ alone. Try $a_n = x^n$.

So we must have, for all $n$: $x^n = x^{n-1} + x^{n-2}$.

This is equivalent to $x^2 = x + 1$. This equation has two solutions, $x = \alpha$ and $x = \beta$:

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2} \quad (1)$$

So both the sequences $(\alpha^n)$ and $(\beta^n)$ satisfy the Fibonacci recurrence relation. But not the initial condition.

**Step 2:** How can we satisfy the initial condition?

Important observation: If $A, B$ are any numbers then the sequence $a_n = A\alpha^n + B\beta^n$ also satisfies the recurrence relation.

So we just need to find $A, B$ in such a way that the initial conditions are satisfied. Taking $n = 0$ and $n = 1$ we get $1 = A + B, 1 = A\alpha + B\beta$. If that is satisfied then it follows that $F_n = a_n$.

A short calculation yields $A = \frac{\alpha}{\sqrt{5}}, B = -\frac{\beta}{\sqrt{5}}$, thus:

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right] \quad (2)$$

This formula seems crazy! The irrational number $\sqrt{5}$ appears at three places, but we know that all the $F_n$ are integers. How does that fit together? Try multiplying out the powers for $n = 1, 2, 3$ and see what happens.

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4 Ansatz = a guessed form of a solution, which has some indeterminates that can be found by plugging in the ansatz into the equation. This German word is also used in mathematical English.

5 If $x \neq 0$; of course $x = 0$ gives a solution but a boring one.

6 Recall how to solve this quadratic equation: $x^2 = x + 1 \iff x^2 - x = 1$, now complete the square: $\iff (x - 1)^2 = 1 \iff x - \frac{1}{2} = \pm \frac{\sqrt{5}}{2} \iff x = \frac{1 \pm \sqrt{5}}{2}$.

7 If you know the general binomial formula explain why the formula always yields a rational number.
Problem 5. Solve the recurrence relation \( a_n = 7a_{n-1} - 12a_{n-2}, \) \( a_0 = 2, \) \( a_1 = 7. \)

Problem 6. Solve the recurrence relation \( a_n = 2a_{n-1} - a_{n-2} \) (a) with initial conditions \( a_0 = a_1 = 1, \) (b) with initial conditions \( a_0 = 0, \) \( a_1 = 1. \)

In this problem you see that the power ansatz does not always work. The reason is that the equation \( x^2 - 2x + 1 = 0 \) has only one solution \( x = 1. \) This gives the solution in (a) but not the solution in (b).

Problem 7. Prove that any solution of the recurrence relation \( a_n = \sqrt{2}a_{n-1} - a_{n-2} \) has period 8, that is, \( a_{n+8} = a_n \) for all \( n. \)

One way to do this is using complex numbers and the power series ansatz: the solutions of \( x^2 - \sqrt{2}x + 1 = 0 \) are \( x = \pm \frac{\sqrt{2}}{2}(1 \pm i). \) Squaring yields \( \pm i, \) and squaring two more times you get 1.

Advanced reading: Some general theory for linear recurrence relations:

A linear recurrence relation is one of the form

\[
a_n = c_{k-1}a_{n-1} + \cdots + c_0a_{n-k},
\]

where \( k \in \mathbb{N} \) and \( c_0, \ldots, c_{k-1} \in \mathbb{R} \) are given. The integer \( k \) is called the length of the recurrence. The Fibonacci recurrence is linear with length 2.

As initial conditions you need the \( k \) numbers \( a_0, \ldots, a_{k-1}. \)

Let us try the power ansatz:

**Step 1:** We look for sequences \( a_n = x^n \) satisfying the recurrence relation. This means \( x^n = c_{k-1}x^{n-1} + \cdots + c_0x^{n-k}. \) Dividing by \( x^{n-k} \) and reordering we see that this is equivalent to

\[
p(x) = 0, \quad \text{where } p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_0.
\]

That is: \( a_n = x^n \) satisfies the recurrence relation if and only if \( x \) is a zero of the polynomial \( p(x). \) This polynomial is called the characteristic polynomial of the recurrence relation.

Background knowledge: a polynomial of degree \( k \) has at most \( k \) distinct zeroes.

**Step 2:** How do we find a sequence satisfying, in addition, the initial conditions? Suppose \( p \) has precisely \( k \) zeroes \( x_1, \ldots, x_k. \) We look for numbers \( A_1, \ldots, A_k \) so that \( a_n = A_1x_1^n + \cdots + A_kx_k^n \) satisfies the initial condition. That is, for given numbers \( a_0, \ldots, a_{k-1} \) we must have

\[
\begin{align*}
a_0 &= A_1 + \cdots + A_k \\
a_1 &= A_1x_1 + \cdots + A_kx_k \\
\vdots \\
a_{k-1} &= A_1x_1^{k-1} + \cdots + A_kx_k^{k-1}
\end{align*}
\]

Another quick way to see this is to use the polar representation of complex numbers and Euler’s formula: \( e^{\pm i\pi/4} = \frac{1}{\sqrt{2}}(1 \pm i) \) and \( e^{\pm 2\pi i} = e^{\pm 2\pi i}. \)
This is a system of \( k \) linear equations for the \( k \) unknowns \( A_1, \ldots, A_k \). One can show that it always has a non-zero solution. \(^9\)

If \( p \) has fewer than \( k \) distinct zeroes then we get \( k \) equations for less than \( k \) unknowns, and this is not solvable for arbitrary given \( a_0, \ldots, a_{k-1} \). Compare Problem 6.

Note that the \( x_i \) and the \( A_i \) can be complex numbers. They may be non-real even if the recurrence relation and the initial conditions are real. See Problem 7 for an example.

**Result:** The linear recurrence relation (3) with arbitrary initial conditions can be solved by the power ansatz if and only if the characteristic polynomial \( p(x) \) in (4) has \( k \) distinct roots.

What do you do if \( p \) has fewer roots? See below!

## 2 Generating functions

Generating functions are an amazingly powerful tool for analyzing sequences.

**Definition 2.1.** The generating function of a sequence \( a_0, a_1, \ldots \) is the function

\[
    f(x) = a_0 + a_1 x + a_2 x^2 + \ldots
\]

that is, the power series with coefficients \( a_0, a_1, \ldots \). \(^{10}\)

Sometimes you can simplify the infinite sum. Fundamental example: \( a_0 = a_1 = \cdots = 1 \).

\[
\text{Geometric series: } 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}
\]

(5)

for \( |x| < 1 \). This follows from the formula for the geometric sum

\[
1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}
\]

by letting \( n \to \infty \) since \( x^{n+1} \to 0 \) if \( |x| < 1 \). \(^{11}\)

Squaring (5) and multiplying out we get \( 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \). You get the same result by differentiating (5).

\(^9\)The coefficient matrix is the so-called Vandermonde matrix, it is always invertible if all \( x_i \) are distinct.

\(^{10}\)If you want to know the domain of definition of \( f \) then you should investigate for which \( x \) the infinite sum converges. However, for many problems (in particular for these notes) this is irrelevant. This statement can be justified rigorously by considering all series as formal power series.

\(^{11}\)Another, formal proof: \( (1 + x + x^2 + \cdots)(1-x) = 1 - x + x - x^2 + x^2 - x^3 \cdots = 1 \).
Problem 8. Find simple expressions for the generating functions of the sequences
\[ a_n = 2^n, \quad b_n = n, \quad c_n = n2^n, \quad d_n = n^2, \quad (e_n) = (1, 0, 1, 0, 1, 0, \ldots) \]

Solution of the Fibonacci recurrence relation via generating functions

Using \( F_n = F_{n-1} + F_{n-2}, \quad F_0 = F_1 = 1 \) we get

\[
f(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 \ldots
= 1 + x + (F_1 + F_0)x^2 + (F_2 + F_1)x^3 + \ldots
= 1 + (x + F_1 x^2 + F_2 x^3 + \ldots) + (F_0 x^2 + F_1 x^3 + \ldots)
= 1 + xf(x) + x^2f(x)
\]

hence

\[
f(x) = \frac{1}{1 - x - x^2} = - \frac{1}{x^2 + x - 1}
\]

Now the trick is to expand this in a power series by a different route:

Step 1: Partial fractions. The zeroes of the polynomial \( x^2 + x - 1 \) are
\( a = \frac{1 + \sqrt{5}}{2} \) and \( b = \frac{1 - \sqrt{5}}{2} \), so we have \( x^2 + x - 1 = (x - a)(x - b) \). We now look for numbers \( C, D \) satisfying

\[
\frac{1}{x^2 + x - 1} = \frac{C}{x - a} + \frac{D}{x - b}.
\]

Multiplying by \( x^2 + x - 1 = (x - a)(x - b) \) we get

\[
1 = C(x - b) + D(x - a) = (C + D)x + (-Cb - Da)
\]

and by comparing coefficients of like powers of \( x \) we get \( 0 = C + D, 1 = -Cb - Da \). After a short calculation we find \( C = \frac{1}{\sqrt{5}} = -D \), so

\[
f(x) = - \frac{1}{\sqrt{5}} \left( \frac{1}{x - a} - \frac{1}{x - b} \right) \quad (6)
\]

Step 2: Using the geometric series we now obtain

\[
\frac{1}{x - a} = \frac{1}{a} \frac{1}{x - a} - 1 = - \frac{1}{a} \frac{1}{1 - \frac{x}{a}} = - \frac{1}{a} \left( 1 + \frac{x}{a} + \frac{x^2}{a^2} + \ldots \right)
\]
The coefficient of \(x^n\) is \(-a^{-n-1}\). The result for \(\frac{1}{x-b}\) is analogous, and using \(6\) we get that the coefficient of \(x^n\) in \(f(x)\) is
\[
\frac{1}{\sqrt{5}} (a^{-n-1} - b^{-n-1})
\]
is. Now this coefficient is also equal to \(F_n\), by the definition of \(f(x)\). So this is the formula for \(F_n\) that we were looking for. (It looks different than \(2\) but is actually the same because of \(\alpha, \beta\).)

**Linear recurrences for which the power ansatz fails**

Using generating functions you can also find the 'missing' solution in Problem \(6\) which we could not find using the power ansatz:

**Problem 9.** Find a closed formula for the solution of \(a_n = 2a_{n-1} - a_{n-2}\), \(a_0 = 0, a_1 = 1\) using generating functions.

**Advanced reading: General theory of linear recurrence relations, part II:**

How do we solve a general linear recurrence relation \(3\) using generating functions? Let \(p(x)\) be the characteristic polynomial as in \(4\). The generating function for the \(a_n\) has the form \(f(x) = \frac{r(x)}{q(x)}\), where \(q(x) = x^k p\left(\frac{1}{x}\right)\) is the polynomial reciprocal to \(p\) and where \(r\) is a polynomial of degree \(< k\) which is determined using the initial conditions. The partial fractions decomposition of \(f\) is a sum of terms of the form \(A x^{n l} x^{n_i}\) where \(z_i\) are the zeroes of \(q\) and where \(l = 0, \ldots, d_i - 1\), with \(d_i\) denoting the multiplicity of the zero \(z_i\). Now you can expand the function \(\frac{1}{(1-x^j)^l}\) as power series with coefficients \(n(n-1)\ldots(n-l+1)\). Note that the latter is a polynomial in \(n\) of degree \(k\). Proceeding as in the Fibonacci example we get:

**Result:** Suppose that the characteristic polynomial \(4\) of the linear recurrence relation \(3\) has the zeroes \(x_1, \ldots, x_m\) with multiplicities \(d_1, \ldots, d_m\), respectively. Then the general solution of the recurrence relation is a sum of terms of the form \(A n^l x^{n_i}\), \(0 \leq l < d_i\), \(i = 1, \ldots, m\). Note that \(p\) having less than \(k\) zeroes is equivalent to \(p\) having multiple zeroes (i.e. some \(d_i\) is bigger than 1).

In short: If \(p\) has multiple zeroes than the power ansatz needs to be extended to include terms of the form \(n^l x^n\), where \(l\) is less than the multiplicity of \(x\) as a zero of \(p\).

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\(^{12}\)It is not a coincidence that \(a = \frac{1}{\alpha}, b = \frac{1}{\beta}\): \(\alpha, \beta\) are the zeroes of \(p(x) = x^2 - x - 1\), the characteristic polynomial of the Fibonacci recurrence. The polynomial \(1 - x - x^2\) which occurred as denominator of \(f(x)\) is reciprocal to \(p(x)\) in the sense that the order of coefficients is reversed, or equivalently \(1 - x - x^2 = x^2 p\left(\frac{1}{x}\right)\). Therefore its zeroes must be \(\frac{1}{\alpha}, \frac{1}{\beta}\). All of this generalizes to any linear recurrence relation. Check it!
3 Partition numbers

Let $p_n$ be the number of partitions of $n \in \mathbb{N}$, i.e. of ways to represent $n$ as a sum of natural numbers, ignoring order. For example, the partitions of $n = 4$ are

$$4, 1 + 3, 2 + 2, 1 + 1 + 2, 1 + 1 + 1 + 1,$$

so $p_4 = 5$. We also define $p_0 = 1$. The partition numbers for $0 \leq n \leq 10$ are $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42$.

There is no obvious closed formula or recurrence relation. But:

**Problem 10.** Show that the generating function of the partition numbers is

$$p(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

Let

$$o_n = \text{the number of partitions of } n \text{ with odd summands}$$
$$d_n = \text{the number of partitions of } n \text{ with distinct summands}$$

**Problem 11.** Find $o_n, d_n$ for $n = 1, \ldots, 8$. Conjecture?

Using generating functions it is not difficult to prove the conjecture (again we set $o_0 := 1, d_0 := 1$):

**Problem 12.** Show that the sequences $(o_n), (d_n)$ have generating functions

$$o(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$
$$d(x) = (1 + x) \cdot (1 + x^2) \cdot (1 + x^3) \cdots$$

**Problem 13.** Prove that $o(x) = d(x)$.

So this implies that $o_n = d_n$ for all $n$. It is not easy to prove this directly from the definition. Try it!

Partition numbers have many more surprising properties. For example one can show that

$$p_n \sim \frac{1}{4n\sqrt{3}} e^{\sqrt{\pi n} \sqrt{2/3}}$$

\[13\]

It is customary to count a sum with just one term also as a partition. In this way all formulas are much nicer than they would be otherwise.
where $\sim$ means that the ratio of the left and right hand sides tends to 1 as $n \to \infty$. Also, the partition numbers satisfy a highly non-trivial (and hard to find!) recurrence relation:

$$p_n = p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + p_{n-12} + p_{n-15} - \cdots$$

(Euler’s pentagonal number theorem)

4 An unsolved problem: The Collatz problem

Define a sequence as follows: Pick a natural number $a_0$ and then let

$$a_{n+1} = \begin{cases} 
\frac{a_n}{2} & \text{if } a_n \text{ is even} \\
3a_n + 1 & \text{if } a_n \text{ is odd}
\end{cases}$$

Let’s try some examples:

- $a_0 = 1$ yields $1, 4, 2, 1, 4, 2, 1, \ldots$
- $a_0 = 3$ yields $3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$

If you try some more values of $a_0$, you will find that sooner or later you will always get 1 (and then 4, 2, 1, \ldots). It is an unsolved problem to prove (or disprove) that this is true for all initial values $a_0$.

5 Mixed problems

Problem 14. A sequence begins 1, 2, 4. What’s the next term?

Problem 15. Let $a_{n+1} = \frac{1}{2} \left( a_n + \frac{1}{a_n} \right)$, $a_0 = 2$. Find a closed formula for $a_n$.

Problem 16. Let $x > 0$. How can you calculate $\sqrt{x}$ quickly to many digits, using only the basic arithmetic operations?

Problem 17. Let $a_0 = a_1 = 1$ and $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$ for $n \geq 2$. What happens for $n \to \infty$?

Problem 18. Find $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$. What does this expression actually mean?

\footnote{To prove this you need complex analysis. See the book Tom Apostol: Introduction to Analytic Number Theory.}
Problem 19. Let $a_0 = 1$, $a_{n+1} = a_n + \frac{1}{a_n}$. Is the sequence $(a_n)$ bounded?

Problem 20. Let $\alpha = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Find $\frac{1}{\sqrt{5}} \alpha^{12}$ to two digits after the decimal point without using a calculator.

Problem 21. In how many ways can you tile a $2 \times n$ rectangle using $1 \times 2$ dominoes?

Problem 22. Show that $p_n$, the number of partitions of $n$, satisfies $p_n \geq 2^\lfloor \sqrt{n} \rfloor$ for $n \geq 2$.

6 Hints

Hint 1. For $c_n$: Add 1, then the recurrence relation changes to $c_n + 1 = 2(c_n - 1 + 2c_n)$. What does this mean for $d_n = c_n + 1$?

Hint 2. Is $(a_n)$ decreasing? The arithmetic-geometric mean inequality (AGM) is useful: $\sqrt{xy} \leq \frac{x+y}{2}$ for $x, y \geq 0$, with equality if and only if $x = y$.

Hint 3. Consider $b_n = a_n - \sqrt{2}$ and estimate $b_{n+1}$ in terms of $b_n$.

Hint 4. For $d_n$ differentiate (5) twice.

Hint 5. Write each factor as a geometric series, then multiply out. In which ways can the term $x^n$ appear?

Hint 6. For $c_n$, Add 1, then the recurrence relation changes to $c_n + 1 = 2(c_n - 1 + 2c_n)$. What does this mean for $d_n = c_n + 1$?

Hint 7. Consider $b_n = a_n - \sqrt{2}$ and estimate $b_{n+1}$ in terms of $b_n$.

Hint 8. For $d_n$ differentiate (5) twice.

Hint 9. Write each factor as a geometric series, then multiply out. In which ways can the term $x^n$ appear?

Hint 10. First calculate some terms of the sequence.

Hint 11. Use that $2^{\lfloor \sqrt{n} \rfloor}$ is the number of subsets of $\{1, 2, \ldots, \lfloor \sqrt{n} \rfloor\}$.

7 Solutions

Solution 1. $a_n = 2n + 1$, $b_n = 3^n$, $c_n = 3 \cdot 2^n - 1$

Solution 2. $a_{n+1} \leq a_n \iff \frac{1}{2}(a_n + \frac{2}{a_n}) \leq a_n \iff \frac{2}{a_n} \leq a_n \iff 2 \leq a_n^2$.

So we need to check whether $a_n^2 \geq 2$ for all $n$. By the AGM inequality $a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-1}}) \geq \sqrt{a_{n-1} \frac{2}{a_{n-1}}} = \sqrt{2}$, so this holds for $n \geq 1$, and since it is true for $a_0 = 5$, the sequence $(a_n)$ decreases for all $n$. Since it is
bounded below, the sequence converges to a limit $x$. The limit must satisfy $x = \frac{1}{2}(x + \frac{2}{x})$, so $x = \sqrt{2}$.

Remark: The initial value is irrelevant, as long as it’s positive. If $a_0 < \sqrt{2}$ then $(a_n)$ decreases only starting at $n \geq 1$.

**Solution 3.** For $b_n = a_n - \sqrt{2}$ we have $b_{n+1} = a_{n+1} - \sqrt{2} = \frac{1}{2}(a_n + \frac{2}{a_n}) - \sqrt{2} = \frac{a_n^2 + 2 - 2\sqrt{2}a_n}{2a_n} = \frac{(a_n - \sqrt{2})^2}{2a_n} < \frac{1}{2}b_n^2$ for all $n$ using $a_n > 1$. So the deviation of $a_n$ from the limit $\sqrt{2}$ gets at least squared and halved in each step. So the number of correct digits after the decimal point roughly doubles in each step.

**Solution 4.** $0, 1, 0, 1, \ldots$ has no limit as $n \to \infty$ (although the fixed point equation $x = 1 - x$ has the solution $x = \frac{1}{2}$).

**Solution 5.** $a_n = 3^n + 4^n$

**Solution 6.** (a) $a_n = 1$ for all $n$. (b) $a_n = n$.

**Solution 7.** Let $a_n = a_n$, $b = a_n + 1$, then $a_{n+2} = \sqrt{2}b - a$, $a_{n+3} = \sqrt{2}a_{n+2} - a_{n+1} = 2b - \sqrt{2}a - b = b - \sqrt{2}a$, $a_{n+4} = \sqrt{2}(b - \sqrt{2}a) - (\sqrt{2}b - a) = -a$, so $a_{n+4} = -a_n$, hence $a_{n+8} = a_n$.

**Solution 8.** $a(x) = \frac{x}{1 - 2x}$, $b(x) = \frac{x^2}{(1-x)^2}$, $c(x) = \frac{2x}{(1-2x)^2}$, $d(x) = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$

**Solution 9.** The generating function is $f(x) = \frac{x}{1-2x+x^2} = \frac{x}{(1-x)^2}$. As in Problem 8 this equals $x + 2x^2 + 3x^3 + \ldots$, so $a_n = n$.

**Solution 11.** For $n = 1, \ldots, 8$ we have $a_n = d_n = 1, 1, 2, 3, 4, 5, 6$.

**Solution 14.** You cannot be sure. One possibility is $1, 2, 4, 8, 16, \ldots$, that is, $a_n = 2^n$. But the formula $a_n = (n^2 + n + 2)/2$ also yields $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$. But $a_3 = 7$.

Suggestion for further study (if you know binomial coefficients): Find the values of \binom{n}{0} + \binom{n}{2} + \binom{n}{4} for $n = 1, \ldots, 5$ and then for $n = 6$. Generalization? Reason?

**Solution 15.** There are various patterns. One way to do it is as follows: notice that in $a_0 = \frac{2}{3}$, $a_1 = \frac{5}{2}$, $a_2 = \frac{41}{30}$ the sum of numerator and denominator is always a power of three: $3, 9, 81$. The exponents are $1, 2, 4$, so $2^n$ (at least for $n = 0, 1, 2$). In addition, the difference between numerator and denominator is $1$. Looking for numbers $N, D$ satisfying $N + D = 3^n$ and $N - D = 1$ we get $N = \frac{1}{2}(3^{2n} + 1)$, $D = \frac{1}{2}(3^{2n} - 1)$. So we guess that $a_n = \frac{3^{n+1} - 1}{3^n} - 1$ for all $n$.

By a short calculation you can check that this satisfies the recurrence relation and initial condition, so it is correct for all $n$.

Suggestion: The core of the calculation is $\frac{1}{2} \left( \frac{x+1}{x-1} + \frac{x-1}{x+1} \right) = \frac{x^2+1}{x^2-1}$. That is, if $a_n = \frac{x+1}{x-1}$ then $a_{n+1} = \frac{x^2+1}{x^2-1}$. Use this to find a closed formula for any
initial value $a_0$.

Solution 16. $a_0 = 1$, $a_{n+1} = \frac{1}{2} \left( a_n + \frac{x}{a_n} \right)$.

Solution 17. The fixed point equation is $x = 2\sqrt{x}$, whose only positive solution is $x = 4$. This suggests to compare $a_n$ to 4. Using induction we get $a_n < 4$ for all $n$. Also $a_{n+1} - a_n = \sqrt{a_n} - \sqrt{a_{n-2}} = (\sqrt{a_n} - \sqrt{a_{n-1}} + (\sqrt{a_{n-1}} - \sqrt{a_{n-2}})$, so inductively $a_{n+1} \geq a_n$ for all $n$. Therefore, $(a_n)$ converges to 4.

Solution 18. It seems obvious that $0 < \sqrt{6} < \sqrt{6 + \sqrt{6}} < \ldots$, so $a_0 < a_1 < a_2 < \ldots$. We need to show that the sequence $(a_n)$ is bounded above. What’s a good candidate for an upper bound? Let us try a solution of the fixed point equation. Solving $x = \sqrt{6} + x$ we get $x = 3$ as unique positive solution. So we first show $a_n < 3$ for all $n$ by induction. Using this we get a formal proof that $a_{n+1} > a_n$. Therefore the sequence converges to 3, i.e. the infinitely nested root has the value 3.

Solution 19. No. If it was bounded then it would have to have a limit since it is obviously increasing. For the limit we would have $x = x + 1$, which is impossible.

Other solution: $a_{n+1} = a_n^2 + 2 + \frac{1}{a_n^2} > a_n^2 + 2$, so $a_n^2 \geq 1 + 2n$ for all $n$. This also shows that the sequence $a_n$ diverges at least like $\sqrt{n}$.

Solution 20. $\frac{1}{\sqrt{5}} \alpha^{12} = F_{12} + \frac{1}{\sqrt{5}} \beta^{12}$ and $F_{12} = 144$, $|\beta| < 0.7 \Rightarrow \beta^2 < 0.49 < \frac{1}{2} \Rightarrow \beta^{12} < \frac{1}{64} \Rightarrow 0 < \frac{1}{\sqrt{5}} \beta^{12} < \frac{1}{100}$, so $\frac{1}{\sqrt{5}} \alpha^{12} = 144.00 \ldots$.

Solution 21. $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1} + a_{n-2}$ (distinguish tilings that start with a vertical domino on the left or with two horizontal dominoes).

So $a_n = F_n$ for all $n$.

Solution 22. For any subset $A = \{a_1, \ldots, a_k\} \subseteq \{1, 2, \ldots, \lfloor \sqrt{n} \rfloor\}$ where $a_1 < \cdots < a_k$ and $k \geq 0$ consider the partition $n = a_1 + \cdots + a_k + r$ where $r = n - a_1 - \cdots - a_k$. The main point is to note that $a_1 + \cdots + a_k \leq 1 + \cdots + \lfloor \sqrt{n} \rfloor = \lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor - 1)/2 \leq \sqrt{n}(\sqrt{n} - 1)/2 < n/2$ so that $r > n/2 \geq \lfloor \sqrt{n} \rfloor$ where the last inequality holds for $n \geq 2$. Therefore the partition $n = a_1 + \cdots + a_k + r$ is written in increasing order, and this shows that any two different subsets $A$ will give different partitions. Therefore $p_n \geq 2^\lfloor \sqrt{n} \rfloor$.

8 Further reading

There are many good books on problem solving. *Arthur Engel's book*(Problem-Solving Strategies) has lots of problems (and hints/solutions)
at all levels (also on sequences). I also like Paul Zeitz’s book (The Art and Craft of Problem Solving). Finally, Daniel Grieser’s (yes, that’s me) book (Exploring Mathematics – Problem-Solving and Proof) introduces many problem-solving techniques (with many explicitly solved problems) and prepares at the same time for university style mathematics.

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15 Arthur Engel trained, very successfully, the German IMO team for many years in the 1970s and 1980s.
16 Paul Zeitz has trained, very successfully, the US IMO team.