Determinants and Even and Odd Permutations

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1. For each $n \geq 3$, find the determinant of the $n \times n$ "multiplication table" and "addition table." For example, the 3×3 multiplication table and addition table, respectively, are

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}.$$

   What about tables formed from products or sums of $2n$ arbitrary real numbers?

2. Find the determinant of the $n \times n$ matrix with x's on the diagonal and y's elsewhere.

3. Prove that if A and D are square matrices, then

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det A \cdot \det D.$$

   (Here C is an appropriately sized matrix and 0 an appropriately sized matrix of 0's.)

4. What does rotating an $n \times n$ matrix by 90° do to the determinant?

5. The nth *Hadamard matrix*, H_n, is a matrix of 1's and -1's defined as follows: H_0 is the 1×1 matrix $[1]$, and for $n \geq 0$,

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

   Evaluate $\det H_n$ (a) by elementary row and/or column operations, (b) geometrically.

6. Given a 2×2 matrix with integer entries and determinant 1, prove that it can be transformed into the 2×2 identity matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ by a sequence of elementary row operations, each consisting of adding one row to the other or subtracting one row from the other. For a challenge, generalize to $n \times n$.

7. (a) Prove that there does not exist a move on the Rubik's Cube that swaps two edge pieces and leaves the rest of the cube fixed.
   (b) Prove that there does not exist a move on the Rubik's Cube that flips over one edge piece and leaves the rest of the cube fixed.

8. (a) When does the operation $x \mapsto -3x$ define an even permutation on the congruence classes mod p, where $p \neq 3$ is prime?
   (b) Prove that -3 is a square mod p iff p is a square mod 3. (One can prove quadratic reciprocity along these lines, as was demonstrated by Zolotarev in 1872.)

9. Let S_n be the set of all permutations on the set $\{1, \dots, n\}$, and let $\operatorname{sgn}(\sigma)$ denote the sign of the permutation σ (1 if σ is even, -1 if σ is odd). Also let $\operatorname{fp}(\sigma)$ denote the number of fixed points of σ, that is, the number of i such that $\sigma(i) = i$. (The first two notations are standard; the last is not.)
   (a) Compute

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x^{\operatorname{fp}(\sigma)}$$

   in terms of x.
   (b) Find the least integer k such that

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \operatorname{fp}(\sigma)^k$$

   is nonzero, and compute it.

Putnam Problems on Determinants

1. (2014 A2) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is
$$\frac{1}{\min(i,j)}$$
for $1 \leq i, j \leq n$. Compute $\det(A)$.

2. (1992 B5) Let D_n denote the value of the $(n-1) \times (n-1)$ determinant with $3, 4, 5, \ldots, n+1$ down the main diagonal and 1's elsewhere. Is the set
$$\left\{ \frac{D_n}{n!} : n \geq 2 \right\}$$
bounded?

3. (2009 A3) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,
$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$
The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

4. (1999 B5) For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

5. (1995 B3) To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \left[\begin{smallmatrix} 8 & 6 \\ 1 & 7 \end{smallmatrix}\right] = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

6. (2016 B4) Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

7. (2008 A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

8. (2002 A4) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position and play continues in turn until the 3×3 matrix is completed with five 1s and four 0s. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

9. (1994 A4) Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

10. (2011 B4) In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

    Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.