14 February 2017 **Olympiad Tricks Part I**

Once you solve a problem, try to extract the *trick* from the solution. What is the thing that you might not see immediately that makes the problem trivial, easy, or even just doable?

Warmup Problems.

(1) Determine the number of trailing zeros in the decimal expansion of $N = 1\,000\,000!$ (one million factorial).

(2) Let $F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ F_5 = 8,\ F_6 = 13, \ldots$ denote the Fibonacci numbers. Prove that, for any $n \geq 1$, the alternating sum

$$\sum_{k=1}^{n} (-1)^k F_k$$

is equal to a Fibonacci number or the negative of a Fibonacci number.

Contest Problems.

(1) Let f be a real-valued function on the plane such that for every square ABCD in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

(2) Let n be a positive odd integer. Prove that the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

(3) Simplify the expression $\dfrac{2}{\log_4 2000^6} + \dfrac{3}{\log_5 2000^6}$.

(4) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

(The argument of cosine is always in radians, not degrees.) Evaluate d_{2017}.

(5) Suppose that x, y, and z are positive real numbers satisfying $xyz = 1$, $x + 1/z = 5$, and $y + 1/x = 29$. Find $z + 1/y$.

(6) Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

(7) Let $N = \sum_{k=1}^{1000} k \left(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor \right)$. Find the remainder when N is divided by 1000.

(8) Show that there exist integers a, b, c, not all zero, with $|a|$, $|b|$, $|c| < 10^6$, such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

(9) Suppose that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, z. Prove that there exists a function g of a single variable such that $f(x, y) = g(x) - g(y)$ for all real numbers x, y.

(10) Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

(11) Show that the number of ways of representing n as an ordered sum of 1s and 2s equals the number of ways of representing $n + 2$ as an ordered sum of integers > 1. For example: $4 = 1 + 1 + 1 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2$ (5 ways) and $6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2$ (5 ways).