BMC	Berkeley Math Circle	Spring 2017	Handout

Pigeonhole Principle(s)

Pigeonhole Principle: If $n+1$ or more objects (pigeons) are distributed into n boxes (pigeonholes), then at least one box contains two ore more of the objects.

Example 1. Among 13 people there are two who have their birthdays in the same month.
Question: What are the pigeons? What are the pigeonholes?

Question 1. There are five married couples. How many of the ten people must be selected to guarantee that a married couple has been selected?
Again think carefully about what the pigeons and pigeonholes are.

Question 1'. There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

Question 2. Given three integers a_1, a_2, a_3 at least one of the consecutive sums must be divisible by three:

$$a_1,\ a_2,\ a_3,\ a_1+a_2,\ a_2+a_3,\ a_1+a_2+a_3.$$

[Hint: remainders !].

Question 2'. Given n integers $a_1, a_2, \dots, a_n$ at least one of the consecutive sums $a_k + a_{k+1} + \cdots + a_\ell$ where $1 \le k \le \ell \le n$ must be divisible by n. Note that we consider a single integer as a consecutive "sum" as well, i.e., $k = \ell$ is okay.

Question 3. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every but, to avoid tiring herself, she decides not to play more than 12 games during any week. Show that there exists a succession of consecutive days during which the chess master will have played *exactly* 21 games.
Hint: Say a_1 is the number of games played on the first day, a_2 on the first and second day, a_3 on the first three days, etc. What can you say about a_{77} ? Then also consider $a_1+21, a_2+21, \dots, a_{77}+21$.

Question 4. From the integers $1, 2, \dots, 200$ we choose 101 integers. Show that, among the integers chosen, there are two such that one of them is divisible by the other.
Hint: every positive integer can be written as $2^k * a$ where $k \ge 0$ and a is an odd integer.

Strong Pigeonhole Principle: Let $q_1, q_2, \dots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are distributed into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, or the third box contains at least q_3 objects, etc.

Can you prove the Strong Pigeonhole Principle ? The Pigeonhole Principle is a special case of the Strong Pigeonhole Principle: what are the $q_1, \dots, q_n$ in this case?

A common special case of the Strong Pigeonhole Principle says the following: Let n and r be positive integers. If $n(r-1)+1$ objects are distributed into n boxes, then at least one of the boxes contains r or more of the objects.

Can you deduce this special case from the Strong Pigeonhole Principle ? What are $q_1, q_2, \dots, q_n$ in this case?

Question 5. A basket of fruit is arranged out of apples, bananas, and oranges. What is the smallest number of fruit that should be put into the basket to guarantee that either there are at least eight apples, or at least six bananas, or at least nine oranges ?

Question 6. Show that if $n+1$ integers are chosen from $1, 2, 3, \dots, 2n$ then there are always two which differ by 1.

Question 7. Show that if $n+1$ integers are chosen from $1, 2, 3, \dots, 3n$ then there are always two which differ by at most 2.

Question 8. Can you generalize Question 6 and Question 7 ?

Question 9. Five kids are standing shoulder to shoulder in a line in the school yard. Show that at least three kids can take a step forward so that their heights, from left to right, are increasing, or their heights are decreasing. Here when we say increasing or decreasing we also include the case that two or more kids can have the same height.
Hint: Suppose there are no three kids whose heights are increasing. Now try to show that there are at least three kids whose heights are decreasing. For each of the five kids, consider the length of the sequence of kids whose heights are increasing *starting with that kid*. Each of these five numbers are either 1 or 2.

Question 10. Repeat the above question where five is replaced with ten, and three is replaced with four.

Question 11 (Challenge). More generally prove the following: Show that every sequence $a_1, a_2, \dots, a_{n^2+1}$ of n^2+1 real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Question 12. Consider a group of six kids. Every pair of these six kids either like each other or hate each other. Show that there must be a group of three kids that is either a love clique or hate clique, i.e., there must exist three kids who mutually all like each other or all hate each other. Can you also argue that if we change six to five, this is not true ?