Lord of the Rings & Fields: ∞ many "number" systems

Berkeley Match Circle, Intermediate II

Chris Overton - week 1: 2016.09.13

Today will focus more on introductions and setup - next week, we'll use this to cover more results.

If you understand the first set of definitions and are bored, you can work on the numbered problems below! Their difficulty is indicated by marks "E","M", and "H".

To get started, we need to know what a *group* is: it has elements $\{g_i\}$, a special "neutral" element 1, and an *associative* binary operation * between two elements. For each element g, there is an "inverse" element g^{-1} such that $g^{-1} * g = g * g^{-1} = 1$.
Note that * might not be commutative, i.e. it may be that $g_i * g_j \neq g_j * g_i$.
For commutative groups, we often write the neutral element as "0" and the operation as "+".

E1) Are the integers $\mathbb{Z}$ a group under addition?
E2) How about the even integers $2\mathbb{Z}$?
E3) How do you "fix" the integers to obtain a group under multiplication?
E4) How do you "fix" the even integers to obtain a group under multiplication?
H5) Find all the groups with numbers of elements up to 8. (That could keep you busy for a while if you haven't done it yet!)

A *ring* R is a set r_i of elements and *two* operations + and $*$, respectively with neutral elements 0 and 1, so that:
a) R is a commutative group under the operation +
b) Without the element 0, R is "almost" a group under the operation $*$, but elements might not have inverses
c) Operations $*$ and + cooperate according to the distributive law: $r_i * (r_j + r_k) = (r_i * r_j) + (r_i * r_k)$. Just as in usual arithmetic, we could have left out the parentheses in that last expression, because * is done before +.

If elements without 0 form a group under $*$, *and* it is commutative, this is called a *field*.
A less popular situation is a "rng" (pronounced 'rung') when there is no neutral element 1 under $*$.

M6) Think of infinitely many different rings.
M+7) Think of infinitely many different *kinds* of infinite sets of rings - actually not that hard...

M+8) For each of the following, determine whether they're ring, a 'rng', a field, or, if none of these, what's missing? (Careful!)
- For a fixed positive integer n, the remainder classes of $\mathbb{Z}$ when "divided" by $n\mathbb{Z}$. This may be written $\mathbb{Z}/n\mathbb{Z}$, or even $\mathbb{Z}/n$.
- Does it make sense to consider + and $*$ operations on $\mathbb{Z}/\frac{1}{2}$? What about $\mathbb{Z}/2\pi$?

Fields are "nicer" than rings, and have interesting structure.
E9) There is exactly one field $\mathbb{F}_3$ with three elements. Work out its addition and multiplication tables.
M10) In the last problem, there is no element f such that $f^2 = -1$. Can you make a field that also contains such an element (call it $\sqrt{-1}$)? If so, work out its addition and multiplication rules.
M11) Work out the addition and multiplication rules for the smallest field that contains the rationals $\mathbb{Q}$ as well as $\sqrt{-1}$.

Vectors and matrices
H12) You may know two kinds of products ($\cdot$ and $\times$) between vectors in three-space $\mathbb{R}^3$. Can you use one of these to make a ring?
M13) Instead of using vectors, do the 3 by 3 matrices $\mathbb{M}_3$ form a ring under addition and multiplication? (...assuming you know how to define these!) If so, do any subsets of these form rings, fields, or rng's?
M14) Say that instead of requiring that both + and $*$ be allowed between matrices, you only allow $*$. What other subsets of $\mathbb{M}_3$ can you think of that are also groups?