# Patterns and Connections

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## Square Numbers and The Pythagorean Theorem

Recall the difference of squares identity: $a^2 - b^2 = (a - b)(a + b)$.

First, we use the identity in the form $a^2 = (a - b)(a + b) + b^2$ to mentally compute the values of squares.
The key question is: what multiple of 10 is nearest our number?

Example $17^2 = (17 + 3)(17 - 3) + 3^2$, which equals $20 \cdot 14 + 9 = 289$.

Example $41^2 = (41 - 1)(41 + 1) + 1^2 = 40 \cdot 42 + 1 = 1681$.

**Exercise 1**: compute the squares of 18, 19, 27, 29, 49, 98, 99.

Next we will use the identity to verify that we have Pythagorean Triples, i.e. numbers $a$, $b$, and $c$ such that $a^2 + b^2 = c^2$.
In these examples we *do not need to know the values of the squares*!

Verify that $20^2 + 21^2 = 29^2$. We compute $29^2 - 21^2 = 8 \cdot 50 = 16 \cdot 25 = 20^2$

1. Show that the following are Pythagorean Triples: 99-101-200, 60-61-11, 8-15-17, 16-63-65.

2. Let $a = m^2 - n^2$, $b = 2\,m\,n$, and $c = m^2 + n^2$. Prove that $a^2 + b^2 = c^2$.

We will show that all pythagorean triples are of the form $m^2 - n^2$, $2\,m\,n$, $m^2 + n^2$ for some pair of integers $m$ and $n$.
This makes finding Pythagorean triples suited to using a multiplication table (provided here for your convenience).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 | 77 | 84 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 | 99 | 108 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 11 | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 | 110 | 121 | 132 |
| 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 144 |

3. Notice that 20-21-29 form the sides of a very nearly isosceles right triangle.
Can you find other nearly isosceles right triangles, namely ones whose legs differ by 1?

## Squangular Numbers

Recall that the $n^{\text{th}}$ triangular number is given by $T_n = \frac{n(n+1)}{2}$.
Here is a list of the first 10 triangular numbers:

1 3 6 10 15 21 28 36 45 55

1. I used the formula $\frac{n(n+1)}{2}$ to make the list
Try to find another way of generating the list.

2. Notice that the triangular numbers 1 and 36 are also perfect squares. Such numbers are called *Squangular*.
Are there other triangular numbers that are squares.

Suppose that the triangular number $T_{n-1} = \frac{n(n-1)}{2}$ is also a perfect square: $\frac{n(n-1)}{2} = m^2$. Here is a way of finding another Squangular number:

$\frac{n(n-1)}{2} = m^2 \Longrightarrow 4\,n^2 - 4\,n = 8\,m^2 \Longrightarrow 4\,n^2 - 4\,n + 1 = 8\,m^2 + 1 \Longrightarrow (2\,n - 1)^2 = 8\,m^2 + 1.$

Then $T_{8\,m^2}$ is a perfect square! Why? $\frac{8\,m^2(8\,m^2+1)}{2} = 4\,m^2(2\,n-1)^2$, which is a perfect square.

3. Given that $T_1$ is a perfect square, verify that $T_8$ is a perfect square, too.
What it the next Squangular number given by $T_{8\,m^2}$?

## $\sqrt{2}$ and Theon's Ladder

Theon's Ladder generates pairs of numbers whose ratios approximate $\sqrt{2}$.
The ladder is shown below.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | 1 | 3 | 7 | 17 | 41 | 99 | 239 | 577 | 1393 | 3363 |
| $b$ | 1 | 2 | 5 | 12 | 29 | 70 | 169 | 408 | 985 | 2378 |

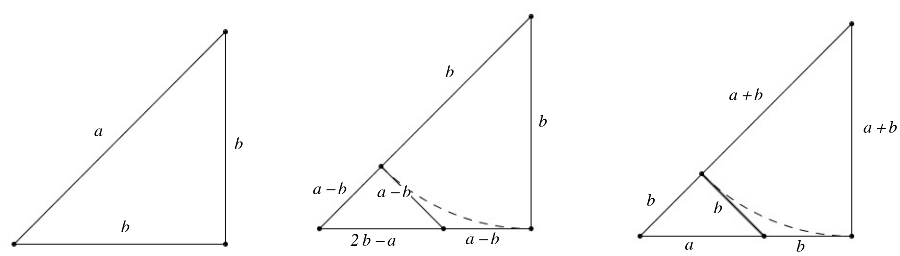
1. I used the recursion relation $a_{n+1} = 2\,a_n + a_{n-1}$ to generate each row.
Try to find a different pattern that generates the ladder.

Notice that in each column $a^2 - 2\,b^2 = \pm 1$.

If $\frac{a}{b} \approx \sqrt{2}$, then $a^2 \approx 2\,b^2$, so the table provides *best* approximations for $\sqrt{2}$ in the sense that the difference $a^2 - 2\,b^2$ is $\pm 1$, which is as small a difference as can be.

2. Prove that $a^2 - 2\,b^2 = \pm 1 \Longrightarrow (a + 2\,b) - 2\,(a + b)^2 = \mp 1$.
(This is just algebra)

3. Use the first two diagrams below to show that $\sqrt{2}$ is irrational: if $\sqrt{2} = \frac{a}{b}$ as in the triangle on the left, then we get a smaller triangle and the approximation $\sqrt{2} = \frac{2\,b-a}{a-b}$.
This leads to what Archimedes called *infinite descent*. Why does this prove that there is no first triangle at all?




4. Show that for $m = a_n \cdot b_n$, then $m^2$ is a Squangular number! For example, $36 = \frac{8 \cdot 9}{2}$.
In the third column we get $5 \cdot 7 = 35$, and $35^2 = 7^2 \cdot 5^2 = \frac{49 \cdot 50}{2}$.
Prove that this always works.
In fact, this gives us *all* Squangular numbers.