Generating Functions and q-analogs

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> A generating function is a clothesline on which we hang a sequence of numbers up for display.
>
> –Herbert Wilf, *Generatingfunctionology*

Generating function basics

Generating functions are a useful tool for solving combinatorial problems.

A classic example of a generating function identity is the geometric series formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

(The term *function* is misleading - here x is just a formal symbol, and the coefficients of the series are the important part!)

Definition. The *(ordinary) generating function* of the sequence $c_0, c_1, c_2, \ldots$ with variable x is the expression

$$c_0 + c_1 x + c_2 x^2 + \cdots .$$

We abbreviate this series as

$$\sum_{i=0}^{\infty} c_i x^i .$$

Generating functions can be added and multiplied together. They can also be differentiated, and sometimes composed!

The following are *definitions* of addition, multiplication, differentiation, and composition of generating functions (we're starting from the beginning here - no calculus allowed.)

- **Addition:** $\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$
- **Multiplication:** $\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$
- **Differentiation:** $\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{i=1}^{\infty} n a_n x^{n-1}$
- **Composition:** If $F(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ and $G(x) = g_1 x + g_2 x^2 + \cdots$, then

  $$F \circ G(x) = \sum_{n \geq 0} f_n G(x)^n = \sum_{N=0}^{\infty} h_N x^N$$

  where

  $$h_N = \sum_{s_1 + \cdots + s_k = N} f_k g_{s_1} g_{s_2} \cdots g_{s_n} .$$

Exercise. Is there a generating function that behaves like an additive identity? A multiplicative identity? Can subtraction and division of generating functions be defined? Why did we only define composition above in the case that G has no constant term?

Exercise. Use the definitions above to prove the generating function identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Notice that $1 - x = 1 - x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$ is a generating function as well.

Tricks for manipulating generating functions

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be generating functions over x. Try your hand at proving the following identities, using only the definitions above.

- $xF(x) = \sum_{n=1}^{\infty} a_{n-1} x^n$
- $\frac{F(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1} x^n$
- $\frac{d}{dx}(F(x) + G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x)$
- $\frac{d}{dx}(F(x)G(x)) = G(x) \cdot \frac{d}{dx}F(x) + F(x) \cdot \frac{d}{dx}G(x)$
- If $b_0 \neq 0$, $G(x)$ has a multiplicative inverse:
$$G(x)^{-1} = b_0^{-1} - b_0^{-1} b_1 x + (b_0^{-3} b_1^2 - b_0^{-2} b_2)x^2 + \cdots .$$
- If $b_0 \neq 0$, $\frac{d}{dx}\left(\frac{F(x)}{G(x)}\right) = \frac{G(x)\frac{d}{dx}F(x) - F(x)\frac{d}{dx}G(x)}{G(x)^2}$.

Exercise. Show that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots .$$

Using generating functions to solve recurrences

Suppose we wish to find an explicit formula for the nth Fibonnacci number F_n, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Consider the generating function

$$G(x) = \sum_{n=0}^{\infty} F_n x^n.$$

Let's manipulate this to take advantage of the recursion:

$$G(x) - xG(x) - x^2 G(x) = F_0 + F_1 x - F_0 x + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n = x.$$

Thus $G(x) = x/(1-x-x^2)$. Using partial fractions and expanding each term as a geometric series, we find that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) x^n,$$

and so $F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$.

Exercise. Use generating functions to find an explicit formula for the nth term in the sequence a_n where $a_0 = 1$, $a_1 = 5$, $a_{n+2} = 4a_{n+1} - 3a_n$.

Problems on Generating Functions

1. Find a formula for the nth term in the sequence a_n where $a_0 = 1$, $a_1 = 5$, $a_{n+2} = 4a_{n+1} - 3a_n$.

2. Find a formula for the nth term of the sequence b_n where $b_0 = 1$, $b_1 = 6$, $b_{n+2} = 4b_{n+1} - 4b_n$.

3. Prove the following generating function identities:

   (a) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

   (b) $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$

   (c) $\frac{1}{1-y(1+x)} = \sum_{k,n} \binom{n}{k} x^k y^n$

4. Simplify $\sum_{n=0}^{\infty} n^2 x^n$. (There are at least three nice ways of doing this!)

5. Prove the following combinatorial identities using generating functions:

   (a) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

   (b) $\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$

   (c) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$

6. Find an explicit formula for

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots$$

   in terms of n.

7. (360 Problems for Mathematical Contests.) Find a closed form expression for the sum

$$S_n = \binom{n}{1} - 3\binom{n}{3} + 5\binom{n}{5} - 7\binom{n}{7} + \cdots.$$

8. (HMMT 2007.) Let S denote the set of all triples (i, j, k) of positive integers satisfying $i + j + k = 17$. Compute

$$\sum_{(i,j,k) \in S} ijk.$$

A variant: q-analogs

A special type of generating function is called a q-analog.

Definition. A *q-analog* of a quantity or formula P is a quantity or formula P_q, such that if we set $q = 1$ or let $q \to 1$ in P_q, we get P.

Example. Let's q-count permutations, according to how "mixed up" they are. A good measure of this is how many *inversions* a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ has, that is, how many pairs (i, j) of entries with $i < j$ have $\pi_i > \pi_j$. We let $\mathrm{inv}(\pi)$ be the number of inversions of π.

If we q-count by adding q^i for each permutation having i inversions, we get the sum

$$\sum_{\pi} q^{\mathrm{inv}(\pi)}$$

over all permutations π. For $n = 3$, we have the permutations 123, 132, 213, 231, 312, and 321, which have 0, 1, 1, 2, 2, and 3 inversions respectively. So our q-count is

$$1 + 2q + 2q^2 + q^3.$$

This polynomial is a q-analog of $n!$, and we write it as $(n)_q!$, pronounced "n q-factorial".

1. **The q-factorial**:
   (a) Our answer in the example above was $(3)_q! = 1 + 2q + 2q^2 + q^3$. Can you factor this polynomial?
   (b) Find a formula for $(n)_q!$ as a product of factors.
   (c) What does this formula suggest we should use as a q-analog of a number n?
   (d) Let $\mathrm{maj}(\pi)$ be the sum of the indices i for which $\pi_i > \pi_{i+1}$. Show that $\sum q^{\mathrm{maj}(\pi)} = (n)_q!$.

2. **q-binomial coefficients**
   (a) Your house is on the corner of a block in a city whose streets are laid out in a grid. Your grandmother's house is also on a corner, and is m blocks east and n blocks north of your house. In how many ways can you travel either east or north along the roads in order to get from your house to your grandmother's?
   (b) Suppose you want to q-count these paths instead, by the number of blocks to the northwest of the path in the $m \times n$ rectangle. In other words, we are summing $q^{A(P)}$ over all paths P to grandma's house, where $A(P)$ is the area above the path in the rectangle. Show that this sum factors as

   $$\frac{(m+n)_q!}{(n)_q!(m)_q!},$$

   where the q-factorial is that of the previous problem. This is called the *q-binomial coefficient*, and is written $\binom{m+n}{n}_q$.

3. **q-identities**
   (a) **q-addition of q-numbers:** Show that if $(n)_q = 1 + q + q^2 + \cdots + q^{n-1}$, then $(a)_q + q^a(b)_q = (a + b)_q$ for any integers a and b.

(b) Using the formula for q-binomial coefficients above, prove the following "q-Pascal's identity".

$$\binom{n}{k-1}_q + q^k \binom{n}{k}_q = \binom{n+1}{k}_q$$

(c) Use this recursion to write out the first few rows of the "q-Pascal Triangle". What do you notice?

(d) Show that the following q-analog of the binomial theorem holds:

$$\prod_{k=0}^{n} (1 + q^k x) = \sum_{m=0}^{n} q^{m(m+1)/2} \binom{n}{m}_q x^m$$

(e) Can you find a q-analog of the Hockey stick identity,

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}?$$

4. **q-real numbers**: If $(n)_q = 1 + q + q^2 + \cdots + q^{n-1}$, how can we define $(\alpha)_q$ for any real number α, in a way that generalizes this formula?

5. **q-Catalan numbers**: Suppose your grandmother's house is at a corner n blocks east and n blocks north of your house. This time, you decide to q-count the number of ways to travel from your house to hers with north or east steps, but always staying to the northwest of the main diagonal on the grid of streets. As with the q-binomial coefficients, you count by q^i if there are i squares to the northwest of your path in the $n \times n$ grid. Show that if $C_n(q)$ is the resulting q-number, then

$$C_{n+1}(q) = \sum_k C_k(q) C_{n-k}(q) \cdot q^{(n-k)(k+1)}.$$