Tantalizing Triangles

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- ***The Pythagorean Theorem***

In a right triangle, the sum of the squares of the sides equals the square of the hypotenuse.

This is perhaps the most famous theorem in elementary mathematics.

1. Let $a = m^2 - n^2$, $b = 2\,m\,n$, and $c = m^2 + n^2$. Prove that $a^2 + b^2 = c^2$.
   Thus the pattern $m^2 - n^2$, $2\,m\,n$, $m^2 + n^2$ always yields a triple of integers whose sides form a right triangle; we call three such numbers a ***Pythagorean Triple***
2. We will prove that all pythagorean triples arise according to this pattern!

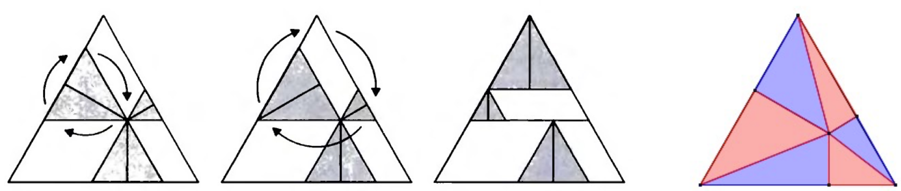
Recall the difference of squares identity: $a^2 - b^2 = (a - b)\,(a + b)$.
We can use this identity to verify that three numbers form a pythagorean triple.

To verify that $20^2 + 21^2 = 29^2$, compute $29^2 - 21^2 = 8 \cdot 50 = 16 \cdot 25 = 20^2$ Done!

3. Show that the following are Pythagorean Triples: 99-101-200, 60-61-11, 8-15-17, 16-63-65.
4. Notice that 20-21-29 form the sides of a very nearly isosceles right triangle.
   Can you find other nearly isosceles right triangles, namely ones whose legs differ by 1?

- **Viviani's Theorem**

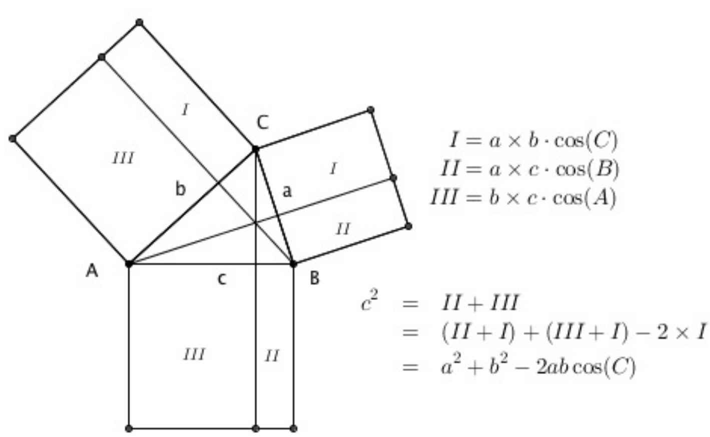
The perpendiculars to the sides, from any point inside an equilateral triangle, add up to the height of the triangle.



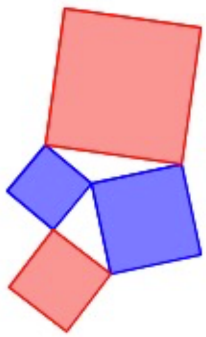
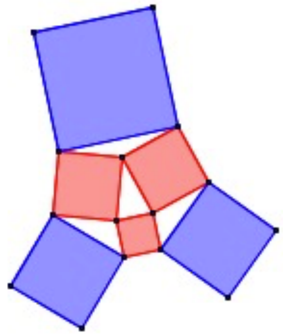
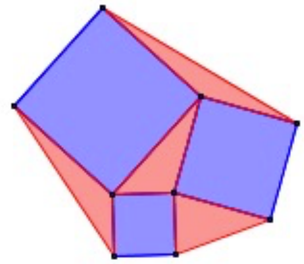
In the second equilateral triangle above, the regions formed by the perpendiculars to the sides are formed;
show that the red area equals the blue area.

- **The Law of Cosines (and Sines) and applications**

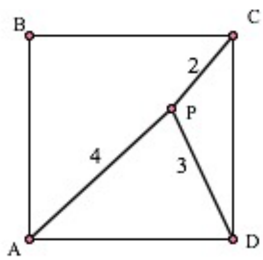
Proof by altitudes: recall that the altitudes of a triangle meet at a point (the Orthocenter of the triangle).
Consider the following areas.

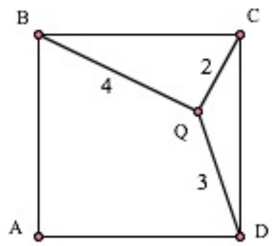


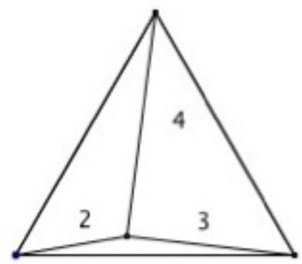
Given the configuration of squares in the first two figures, find a simple relationship between the red and blue areas.
In the third figure, prove that the red areas are all equal.

Find the side of an equilateral triangle containing a point with distances 2, 3, and 4 to the vertices (this problem has one solution)
Find the side of a square containing a point with distances 2, 3, and 4 to the vertices (this problem has two solutions).







- **Triangles with 60 ° or 120 ° angles.**

  Prove that a 3-5-7 triangle has a 120° angle. Use this fact to find an integer triangle with a 120° angle.

  Recalling that $\cos(60°) = \frac{1}{2}$ and $\cos(120°) = -\frac{1}{2}$ we find that triangles obeying $c^2 = a^2 + b^2 \pm a\,b$ have $\angle C = 60°$ (+) or $\angle C = 120°$ (−).

- **Triangles ΔABC with ∠C = 2 ∠A.**

  Show that a 4-5-6 triangle has one angle exactly twice another.
  (Hint: use the Law of Cosines and the identity $\cos(2x) = 2\cos^2(x) - 1$)
  Finding the Pattern
  Suppose that $\angle C = 2\angle A$. The Law of Sines tells us that $\frac{\sin A}{a} = \frac{\sin C}{c}$
  Use the identity $\sin(2x) = 2\sin(x)\cos(x)$ and algebra to arrive at the simple relation $c^2 = a\cdot(a+b)$.

  Find other triangles with angles in a two-to-one ratio.

- **Heron's Formula —Integer Triangles with Integer Areas**

  Recall that the area of a triangle with sides a, b, and c is given by $A^2 = s(s-a)(s-b)(s-c)$ where $s = \frac{a+b+c}{2}$.

  Show that a 13-14-15 triangle has integer area. What about 4-13-15?

  Are there any other triangles with consecutive integer sides that also have integer area?