Berkeley Math Circle

September 8, 2015

1. Let G be a graph with n vertices, m edges and the degrees of the n vertices are d_1, $d_2, \cdots d_n$. Prove that $\sum_{i=1}^{n} d_i = 2m$.

2. For any graph G, let $\Delta(G)$ be the maximum degree amongst the vertices in G. Characterize all graphs with $\Delta(G) = 2$.

3. Let $G = (V, E)$ be a graph. The **complement** $\overline{G}$ of G is a graph with the same vertex set as G and $E(\overline{G}) = \{e \notin E(G)\}$ i.e. G has edges exactly where there are no edges in G.

   Let G be a disconnected graph. Prove that its complement $\overline{G}$ is connected.

4. Let G be a connected graph. Prove that two paths which are both a longest path in the graph, contain at least one vertex in common.

5. Let G be a connected graph. An edge e is said to be a **cut-edge** if its removal disconnects the graph. Prove that e is not a cut-edge if and only if e is an edge of a cycle.

6. A graph is said to be **planar** if it can be drawn such that a pair of edges can only cross at a vertex.

   (a) Suppose a (simple) planar graph G has $n \geq 3$ vertices. Prove that G has at most $3n - 6$ edges.

   (b) Show that if a simple planar graph G does not have any 3 cycles, then G has at most $2n - 4$ edges.

   (c) Convince yourself that K_5 (complete 5-graph) and $K_{3,3}$ (complete $(3,3)$-bipartite graph) are not planar.

7. A graph G is said to be **bipartite** if its set of vertices $V(G)$ can be partitioned into two non-empty disjoint sets A, B such that no edge has both endpoints in the same set. Prove that a graph is bipartite if and only if it does not contain an odd cycle.

8. Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are trying to color the computers and cables such that no two computers have the same color, no two cables joined to a common computer have the same color, and no computer is assigned the same color as any cable joined to it. Prove that this can be done using n colors.

9. Given a graph G, let $\chi(G)$ be the minimum number of colors required to color the vertices of G such that no two adjacent vertices are assigned the same color. Let e be the number of edges in G. Prove that

$$\chi(G) \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}.$$

10. Find the smallest positive integer n such that in any set of n irrational numbers, there are three numbers such that the sum of every two of them is again irrational.

11. There are n people in a room. Any group of $m(\geq 3)$ people in the room have a unique common friend. Can you determine n in terms of m?