1. Introduction. Given a triangle, a cevian is a line segment from a vertex to a point on the interior of the opposite side. (The ‘c’ is pronounced as ‘ch’). Figure 1 illustrates two cevians $AD$ and $CE$ in $\triangle ABC$. Cevians are named for mathematician Giovanni Ceva who used them to prove his famous theorem.

Here is a geometry problem involving cevians. Later on, we’ll solve it using mass point geometry.

**Problem 1.** In $\triangle ABC$, shown in Figure 1, side $BC$ is divided by $D$ in a ratio of 5 to 2 and $BA$ is divided by $E$ in a ratio of 3 to 4. Find the ratios in which $F$ divides the cevians $AD$ and $CE$, i.e., find $EF : FC$ and $DF : FA$.

**Archimedes’ Principle of Levers.** Mass point geometry is based on the idea of a seesaw with masses at each end. The seesaw will balance if the product of the mass and its distance to the fulcrum is the same for each mass. For example, if a baby elephant of mass $100$ kg is $0.5$ m from the fulcrum, then an ant of mass $1$ g must be located $50$ km on the other side of the fulcrum for the seesaw to balance.

$$\text{distance} \times \text{mass} = 100 \text{ kg} \times 0.5 \text{ m} = 100,000 \text{ g} \times 0.0005 \text{ km} = 1 \text{ g} \times 50 \text{ km}.$$
2. The objects of mass point geometry. We begin coordinate geometry by defining basic objects like point and line. In this vein, we define the main objects of mass point theory.

**Definition 1.** A mass point is a pair \((n, P)\), also written \(nP\), consisting of a positive real number \(n\), the mass, and a point \(P\) in the plane.

**Definition 2.** We say two mass points coincide, \(nP = mQ\), if and only if \(n = m\) and \(P = Q\), i.e., they correspond to the same ordinary point with the same assigned mass.

**Definition 3 (Addition).** Given two mass points \(nE\) and \(mA\), if \(E \neq A\), we set their sum to be \(nE + mA = (n + m)F\) where \(F\) lies on segment \(EA\) and \(EF : FA = m : n\). If \(E = A\), we set \(nE + mA = (n + m)E\). In either case, the sum is called the center of mass of the two mass points \(nE\) and \(mA\).

**Definition 4 (Scalar Multiplication).** Given a mass point \((n, P)\) and real number \(m > 0\), called a scalar, we define \(m(n, P) = (mn, P)\).

3. Basic Properties of Mass Point Addition and Scalar Multiplication. These mass point operations satisfy the following properties.

**Property 1 (Closure).** The addition of two mass points produces a unique sum, which is also a mass point.

**Property 2 (Commutativity).** \(nP + mQ = mQ + nP\).

**Property 3 (Associativity).** \(nP + (mQ + kR) = (nP + mQ) + kR = nP + mQ + kR\).

**Property 4 (Distributivity).** \(k(nP + mQ) = knP + kmQ\).

**Property 5 (Subtraction).** If \(n > m\) then \(nP = mQ + xX\) may be solved for the unique unknown mass point \(xX\). Namely, \(xX = (n - m)R\) and either \(P = Q = R = X\) or \(P\) is on segment \(RQ\) so that \(RP : PQ = m : (n - m)\).

![Figure 3: Subtracting mass points.](image)

**Exercise 1.** Given mass points \(3Q\) and \(5P\), find the location and mass of their difference \(5P - 3Q\).

**Solution:** By the definition of subtraction, \(5P - 3Q = (5 - 3)R\), where \(P\) is the balancing point of the mass points \(3Q\) and \(2R\). This means \(3|QP| = 2|PR|\), so that \(R\) will be on the other side of \(P\) at a distance of \(\frac{3}{2}|QP|\). \(\square\)

**Solution to Problem 1:** In order to make \(D\) the balancing point of \(BC\), let’s assign a mass of 2 to \(B\) and a mass of 5 to \(C\). To have \(E\) as the balancing point of \(BA\), we assign \(2 : 3/4 = 3/2\) to \(A\). Then at the balancing points on the sides of the triangle, we have \(2B + 5C = 7D\) and \(2B + \frac{5}{2}A = \frac{7}{2}E\).

The center of mass \(8.5X\) of the system \(\{\frac{3}{2}A, 2B, 5C\}\) is located at the sum \(\frac{3}{2}A + 2B + 5C\). The latter can be calculated in two ways according to our associativity property:

\[
\frac{7}{2}E + 5C = \left(\frac{3}{2}A + 2B\right) + 5C = 8.5X = \frac{3}{2}A + \left(2B + 5C\right) = \frac{3}{2}A + 7D.
\]

This implies that \(X\) is located simultaneously on \(EC\) and on \(AD\), so \(X\) must coincide with intersection point \(F\) of the two cevians. Hence \(F\) is the fulcrum of the seesaw balancing \(\frac{3}{2}A\) and \(7D\) and of the seesaw balancing \(5C\) and \(\frac{7}{2}E\). This means that

\[
DF : FA = 3/2 : 7 = 3 : 14
\]

and \(EF : FC = 5 : 7/2 = 10 : 7\).
Problem Solving Technique. Assign masses at the vertices of $\triangle ABC$ in such a way that the intersection point $F$ of the cevians becomes the center of mass of the resulting system. This allows for calculations based on the seesaw principle and our five properties of mass points.

Exercise 2 (Warm-up). If $G$ is on $BY$, find $x$ and $BG:GY$ provided
(a) $3B + 4Y = xG$;  (b) $7B + xY = 9G$

Exercise 3. In $\triangle ABC$, $D$ is the midpoint of $BC$ and $E$ is the trisection point of $AC$ nearer to $A$ (i.e., $AE:EC = 1:2$). Let $G = BE \cap AD$. Find $AG:GD$ and $BG:GE$.

Exercise 4 (East Bay Mathletes 1999). In $\triangle ABC$, $D$ is on $AB$ and $E$ is on $BC$. Let $F = AE \cap CD$, $AD = 3$, $DB = 2$, $BE = 3$, and $EC = 4$. Find $EF:FA$ in lowest terms.

Exercise 5. Show that the medians of a triangle are concurrent in a point which divides each median in a ratio of 2:1 counted from the vertices.

Hint: Assign a mass of 1 to each vertex, as shown below.

Exercise 6 (Varignon’s Theorem). Show that the four midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

4. Splitting Masses. Let’s take a look at another problem. Once we have mastered its solution, we can apply the mass-splitting technique used here to answer a whole new class of questions.

Problem 2. In Figure 6, transversal $ED$ joins points $E$ and $D$ on the sides of $\triangle ABC$ so that $AE:EB = 4:3$ and $CD:DB = 2:5$. Cevian $BG$ divides $AC$ in a ratio of $3:7$ counted from vertex $A$ and intersects the transversal $ED$ at point $F$. Find the ratios $EF:FD$ and $BF:FG$. 
Solution to Problem 2: The mass point property \((m + n)P = mP +nP\) is the basis for splitting masses, the technique to use when dealing with transversals. Here’s how it works. We start by assigning 4 to \(B\) and 3 to \(A\) to balance \(AB\) at \(E\). Then to balance \(AC\) at \(G\), we assign \(\frac{9}{7}\) to \(C\). To balance \(BC\) at point \(D\), \(\frac{18}{35}\) \(B\) is needed. So we now have \((4 + \frac{18}{35})B\). This gives \(\frac{44}{5}\) \(F\) as the center of mass at \(A\), \(B\), and \(C\).

Applying the commutative and associative properties, we obtain:

\[
\frac{30}{7}G + (4 + \frac{18}{35})B = (3A + \frac{9}{7}C) + (4B + \frac{18}{35}B) = (3A + 4B) + (\frac{18}{35}B + \frac{9}{7}C) = 7E + \frac{9}{7}D
\]

This shows that the center of mass lies on both \(ED\) and \(BG\), i.e., it is located at point \(F\). The sought-after ratios can now be read directly from the diagram: \(EF : FD = 9/5 : 7 = 9 : 35\) and \(BF : FG = 30/7 : 158/35 = 75 : 79\).

Ready to give mass-splitting a try? Here are two more examples.

Exercise 7. In \(\triangle ABC\), let \(E\) be on \(AB\) such that \(AE : EB = 1 : 3\), \(D\) on \(BC\) such that \(BD : DC = 2 : 5\), and \(F\) on \(ED\) such that \(EF : FD = 3 : 4\). Finally, let ray \(BF\) intersect \(AC\) at \(G\). Find \(AG : GC\) and \(BF : FG\).

Exercise 8. With the same configuration as in Exercise 7, \(AE : EB = 3 : 1\), \(BD : DC = 4 : 1\), and \(EF : FD = 5 : 1\). Show that \(AG : GC = 4 : 1\) and \(BF : FG = 17 : 7\).

Want more practice? Check out the Art of Problem Solving website’s page on mass points, which includes a list of several AMC and AIME problems that can be solved with mass point geometry.