Mass Point Geometry

Excerpts of an article* by Tom Rike

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1. Introduction. Given a triangle, a *cevian* is a line segment from a vertex to a point on the interior of the opposite side. (The 'c' is pronounced as 'ch'). Figure 1 illustrates two cevians AD and CE in $\triangle ABC$. Cevians are named for mathematician Giovanni Ceva who used them to prove his famous theorem.

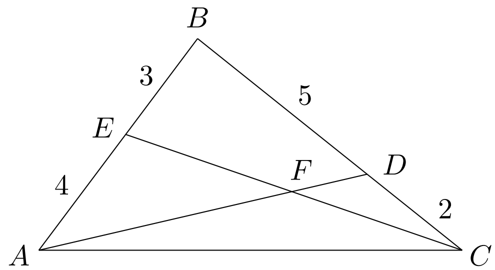

Figure 1: Cevians AD and CE in $\triangle ABC$.

Here is a geometry problem involving cevians. Later on, we'll solve it using mass point geometry.

Problem 1. In $\triangle ABC$, shown in Figure 1, side BC is divided by D in a ratio of 5 to 2 and BA is divided by E in a ratio of 3 to 4. Find the ratios in which F divides the cevians AD and CE, i.e., find $EF : FC$ and $DF : FA$.

Archimedes' Principle of Levers. Mass point geometry is based on the idea of a seesaw with masses at each end. The seesaw will balance if the product of the mass and its distance to the fulcrum is the same for each mass. For example, if a baby elephant of mass 100 kg is 0.5 m from the fulcrum, then an ant of mass 1 g must be located 50 km on the other side of the fulcrum for the seesaw to balance.

$$\text{distance} \times \text{mass} = 100 \text{ kg} \times 0.5 \text{ m} = 100{,}000 \text{ g} \times 0.0005 \text{ km} = 1 \text{ g} \times 50 \text{ km}.$$

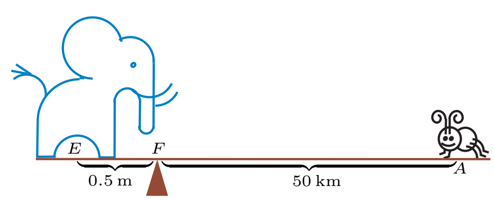

Figure 2: An elephant and an ant balance on a seesaw (Artwork by Zvezda).

*For the full article, see chapter 7 of *A Decade of the Berkeley Math Circle,* Vol. 1, edited by Zvezdelina Stankova and Tom Rike.

2. The objects of mass point geometry. We begin coordinate geometry by defining basic objects like *point* and *line*. In this vein, we define the main objects of *mass point* theory.

Definition 1. A mass point is a pair (n, P), also written nP, consisting of a positive real number n, the mass, and a point P in the plane.

Definition 2. We say two mass points *coincide*, $nP = mQ$, if and only if $n = m$ and $P = Q$, i.e., they correspond to the same ordinary point with the same assigned mass.

Definition 3 (Addition)**.** Given two mass points nE and mA with $E \neq A$, we set their *sum* to be $nE + mA = (n+m)F$ where F lies on segment EA and $EF : FA = m : n$. If $E = A$, we set $nE + mE = (n+m)E$. In either case, the sum is called the *center of mass* of the two mass points nE and mA.

Definition 4 (Scalar Multiplication)**.** Given a mass point (n, P) and real number $m > 0$, called a *scalar*, we define $m(n, P) = (mn, P)$.

3. Basic Properties of Mass Point Addition and Scalar Multiplication. These mass point operations satisfy the following properties.

Property 1 (Closure)**.** The addition of two mass points produces a unique sum, which is also a mass point.

Property 2 (Commutativity)**.** $nP + mQ = mQ + nP$.

Property 3 (Associativity)**.** $nP + (mQ + kR) = (nP + mQ) + kR = nP + mQ + kR$.

Property 4 (Distributivity)**.** $k(nP + mQ) = knP + kmQ$.

Property 5 (Subtraction)**.** If $n > m$ then $nP = mQ + xX$ may be solved for the unique unknown mass point xX. Namely, $xX = (n-m)R$ and either $P = Q = R = X$ or P is on segment RQ so that $RP : PQ = m : (n-m)$.

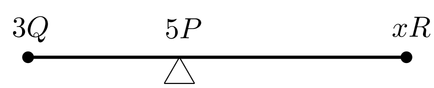

Figure 3: Subtracting mass points.

Exercise 1. Given mass points $3Q$ and $5P$, find the location and mass of their difference $5P - 3Q$.

Solution: By the definition of subtraction, $5P - 3Q = (5-3)R$, where P is the balancing point of the mass points $3Q$ and $2R$. This means $3|QP| = 2|PR|$, so that R will be on the other side of P at a distance of $\frac{3}{2}|QP|$. □

Solution to Problem 1: In order to make D the balancing point of BC, let's assign a mass of 2 to B and a mass of 5 to C. To have E as the balancing point of BA, we assign $2 \cdot 3/4 = 3/2$ to A. Then at the balancing points on the sides of the triangle, we have $2B+5C = 7D$ and $2B+\frac{3}{2}A = \frac{7}{2}E$. The center of mass $8.5X$ of the system $\{\frac{3}{2}A, 2B, 5C\}$ is located at the sum $\frac{3}{2}A + 2B + 5C$. The latter can be calculated in two ways according to our associativity property:

$$\tfrac{7}{2}E + 5C = (\tfrac{3}{2}A + 2B) + 5C = 8.5X = \tfrac{3}{2}A + (2B + 5C) = \tfrac{3}{2}A + 7D.$$

This implies that X is located simultaneously on EC and on AD, so X must coincide with intersection point F of the two cevians. Hence F is the fulcrum of the seesaw balancing $\frac{3}{2}A$ and $7D$ and of the seesaw balancing $5C$ and $\frac{7}{2}E$. This means that

$$DF : FA = 3/2 : 7 = 3 : 14$$
$$\text{and } EF : FC = 5 : 7/2 = 10 : 7.$$

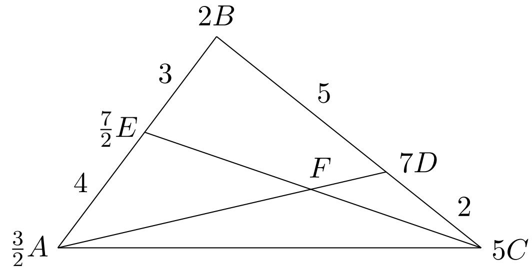

Figure 4: Two cevians problem solved

Problem Solving Technique. Assign masses at the vertices of $\triangle ABC$ in such a way that the intersection point F of the cevians becomes the center of mass of the resulting system. This allows for calculations based on the seesaw principle and our five properties of mass points.

Exercise 2 (Warm-up)**.** If G is on BY, find x and $BG : GY$ provided
(a) $3B + 4Y = xG$; (b) $7B + xY = 9G$

Exercise 3. In $\triangle ABC$, D is the midpoint of BC and E is the trisection point of AC nearer to A (i.e., $AE : EC = 1 : 2$). Let $G = BE \cap AD$. Find $AG : GD$ and $BG : GE$.

Exercise 4 (East Bay Mathletes 1999)**.** In $\triangle ABC$, D is on AB and E is on BC. Let $F = AE \cap CD$, $AD = 3$, $DB = 2$, $BE = 3$, and $EC = 4$. Find $EF : FA$ in lowest terms.

Exercise 5. Show that the medians of a triangle are concurrent in a point which divides each median in a ratio of $2 : 1$ counted from the vertices.

HINT: Assign a mass of 1 to each vertex, as shown below. ◊

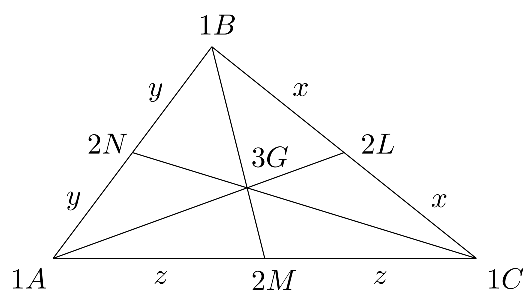

Figure 5: The medians of a triangle are concurrent.

Exercise 6 (Varignon's Theorem)**.** Show that the four midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

4. Splitting Masses. Let's take a look at another problem. Once we have mastered its solution, we can apply the mass-splitting technique used here to answer a whole new class of questions.

Problem 2. In Figure 6, transversal ED joins points E and D on the sides of $\triangle ABC$ so that $AE : EB = 4 : 3$ and $CD : DB = 2 : 5$. Cevian BG divides AC in a ratio of $3 : 7$ counted from vertex A and intersects the transversal ED at point F. Find the ratios $EF : FD$ and $BF : FG$.

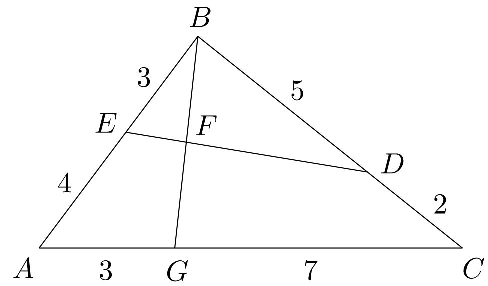

Figure 6: Using mass points with a transversal.

SOLUTION TO PROBLEM 2: The mass point property $(m+n)P = mP + nP$ is the basis for splitting masses, the technique to use when dealing with transversals. Here's how it works. We start by assigning 4 to B and 3 to A to balance AB at E. Then to balance AC at G, we assign $\frac{9}{7}$ to C. To balance BC at point D, $\frac{18}{35}B$ is needed. So we now have $(4 + \frac{18}{35})B$. This gives $\frac{44}{5}F$ as the center of mass at A, B, and C.

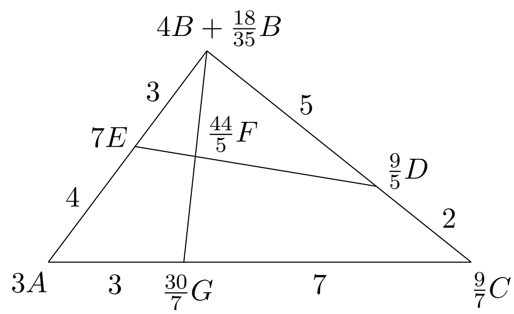

Figure 7: Splitting masses to solve problem 2.

Applying the commutative and associative properties, we obtain:

$$\begin{aligned}\frac{30}{7}G + (4 + \tfrac{18}{35})B &= (3A + \tfrac{9}{7}C) + (4B + \tfrac{18}{35}B) \\ &= (3A + 4B) + (\tfrac{18}{35}B + \tfrac{9}{7}C) = 7E + \tfrac{9}{5}D\end{aligned}$$

This shows that the center of mass lies on both ED and BG, i.e., it is located at point F. The sought-after ratios can now be read directly from the diagram: $EF : FD = 9/5 : 7 = \mathbf{9} : \mathbf{35}$ and $BF : FG = 30/7 : 158/35 = \mathbf{75} : \mathbf{79}$. □

Ready to give mass-splitting a try? Here are two more examples.

Exercise 7. In $\triangle ABC$, let E be on AB such that $AE : EB = 1 : 3$, D on BC such that $BD : DC = 2 : 5$, and F on ED such that $EF : FD = 3 : 4$. Finally, let ray BF intersect AC at G. Find $AG : GC$ and $BF : FG$.

Exercise 8. With the same configuration as in Exercise 7, $AE : EB = 3 : 1$, $BD : DC = 4 : 1$, and $EF : FD = 5 : 1$. Show that $AG : GC = 4 : 1$ and $BF : FG = 17 : 7$.

Want more practice? Check out the Art of Problem Solving website's page on mass points, which includes a list of several AMC and AIME problems that can be solved with mass point geometry.

http://www.artofproblemsolving.com/wiki/index.php/Mass_points