Constructions

Inspired by Chapter 1 of *A Decade of the Berkeley Math Circle, Volume 2.*

October 6, 2015

1. Introduction

Geometry is the mathematics of shape, and so it is a subject that is best understood with the help of pictures. Yet the figures and diagrams that we make with our hands are themselves determined by the rules of geometry. With the help of a straightedge, anyone can draw a straight line. The compass allows us not only to draw perfect circles and arcs, but to measure and duplicate lengths. What can we make with our lines and circles? In this session, we will investigate the geometry of constructions. To give us some motivation, let's consider the following problem.

Problem 1 (Farmer and Cow). During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible?

2. Making Straight Lines

Geometers use a straightedge to draw straight lines by hand with extreme precision. Here's another easy way to produce a straight line by hand:

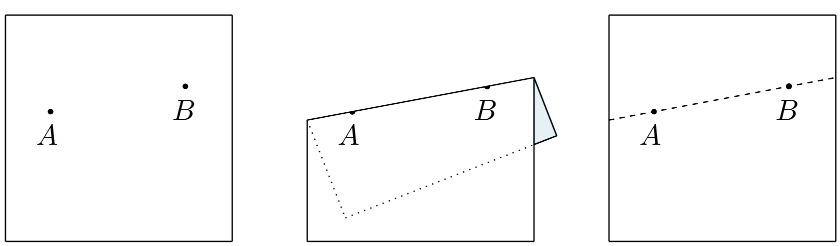

Figure 1: Folding a straight line through points A and B.

Exercise 1. Use your straightedge to draw a line passing through two points given on a sheet of paper. How can you be certain that your line is really straight? Try flipping the straightedge upside down.

Exercise 2. Fold a sheet of paper and, using the previous problem, check that the edge is straight. Can you explain why the edge of a folded paper is straight?

3. Perpendicular Lines

Using only a straightedge and a compass, show that it is possible to construct the objects below. Give an *algorithm* for each construction and prove that it does what it is supposed to do.

Construction 1. Given a segment BC, find its midpoint M.

Construction 2. Given a point S on a line ℓ, erect a perpendicular segment PS to ℓ.

Construction 3. Given line ℓ and point P not on ℓ, drop a perpendicular segment from P to ℓ.

Knowing how to perform a construction is one thing, but understanding why it works is the greater challenge. Also, pay close attention to similarities between two like constructions. One construction may build on the methods of another.

An alternate but very intuitive way to approach constructions 1-3 uses paper folding. For example, if you are given a segment BC drawn on paper, you can *fold* the segment in half, laying point B directly on top of point C. Press a crease into the paper and open.

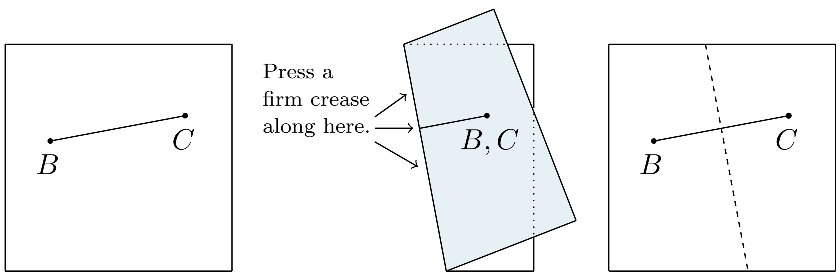

Figure 2: Folding the perpendicular bisector of segment BC.

Construction 4. Given a point P and a line ℓ, use a straightedge and compass to find point Q so that line ℓ perpendicularly bisects segment PQ. Point Q is called the *reflection* of point P across line ℓ, and we say that points P and Q are *symmetric* about line ℓ.

Problem 2. Find the reflection of $(4, 0)$ about the line $y = 2x$.

Problem 3. Lines ℓ_A and ℓ_B are parallel and point M does not lie on either line.
(a) Find the reflection A of M across ℓ_A and the reflection B of M across ℓ_B.
(b) If d is the shortest distance between ℓ_A and ℓ_B, prove that $AB = 2d$.
Consider the case when M is between the two parallel lines and the case when it is not between them.

Speaking of reflection, let's go back to the farmer-and-cow problem. Suppose we make a little change to the problem. If the farmer and the cow were on *opposite* sides of the river, what would be the shortest path between the farmer and the cow? Let's assume that the river has no width; it is just a straight line. If we place the farmer in an equivalent starting position, but on the opposite side of the river from the cow, we are *reflecting* the farmer across the river. This new problem is easier to solve. The question is, is this new problem equivalent to the original one?

4. Isosceles Triangles

A triangle that has two congruent sides is called *isosceles*. On a blank piece of paper, mark a point A. Use your compass to draw a large circle (or a circular arc) centered at A. Randomly choose two points B and C on this circle and draw triangle ABC. What fact from geometry guarantees that $AB = AC$, proving that triangle ABC is isosceles?

The congruent sides of an isosceles triangle are called its *legs*. The angle formed by the legs is the *vertex angle*. The side opposite the vertex angle is called the *base* of the isosceles triangle. Now fold your isosceles triangle in half, laying side AB directly on top of AC. Press a sharp crease into the paper and then open. What is significant about the line you've just folded? See if you can use your folded isosceles triangle to understand the following theorems.

Theorem 1. *In an isosceles triangle, the same line that perpendicularly bisects the base also bisects the vertex angle.*

Theorem 2. *In an isosceles triangle, the base angles are congruent.*

I like to think of both of these theorems as results of the symmetry of isosceles triangles. If M is the midpoint of the base BC of your isosceles triangle, then line AM is the perpendicular bisector of side BC. This means B and C are symmetric about line AM. We might call line AM the *axis of symmetry* of the triangle. This line divides triangle ABC into two triangles, AMB and AMC, which are reflections of one another about line AM. When we fold triangle ABC along line AM, triangles AMB and AMC coincide, and so they are congruent triangles.

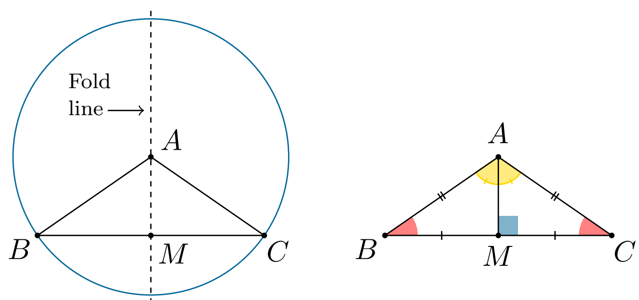

Figure 3: The axis of symmetry of an isosceles triangle divides it into two congruent halves.

Here is another method for constructing an isosceles triangle, starting with its base. Suppose you were given a segment BC and asked to construct isosceles triangle ABC with $AB = AC$. You could start by folding the perpendicular bisector of segment BC as shown in figure 2. Choose a point A on the perpendicular bisector, and then draw sides AB and AC. To understand why this method works, consider the following theorem.

Theorem 3. *A point lies on the perpendicular bisector of a segment if and only if that point is equidistant from the endpoints of the segment.*

Like any "if and only if" theorem, the above statement is two theorems in one. On the one hand, it tells us that the vertex A of isosceles triangle ABC lies on the perpendicular bisector of its base BC, something we already knew from theorem 1. On the other hand, theorem 3 tells us that every point on the perpendicular bisector of segment BC is equidistant from B and C. In the above construction, we chose A to be on this perpendicular bisector, so $AB = AC$.

5. The Bisector of an Angle

The ray which divides a given angle into halves is called the *bisector* of the angle. If you are given angle ABC drawn on paper, you can *fold* the angle in half, laying ray BA directly on top of ray BC. Press a crease into the paper and open. This fold *bisects* angle ABC because it divides the angle into two equal parts.

Construction 5. Given an angle ABC, use a straightedge and compass to draw its angle bisector BL, that is, ray BL such that $\angle ABL = \angle CBL$.

One way to proceed is to construct an isosceles triangle with angle ABC as its vertex angle. Then we merely construct the perpendicular bisector of the base of this triangle. Recall that theorem 1 guarantees that this line also bisects the vertex angle of the triangle.

As an alternative, consider the following process for bisecting an angle ABC: First mark M on BA and P on BC so that $MB = PB$, then mark new points N on BA and Q on BC so that $NB = QB$. Let L be the intersection of MQ and NP. Prove that segment BE is the desired angle bisector.

Exercise 3. Construct a right angle, a 45°-angle, and a 22.5°-angle.

6. Parallel Lines

There are many ways to perform the following construction. How would *you* do it?

Construction 6. Given line ℓ and point A not on ℓ, draw a line n through A parallel to ℓ.

Set the problem up by drawing an arbitrary line ℓ and an arbitrary point A, not on the line. It may occur to you to begin by first constructing a line p through A that is *perpendicular* to ℓ. Then you can construct a second line n through A that is perpendicular to p. It's a good strategy with compass and straightedge, and it's even easier to do with paper folding. Try this strategy now.

Here's a second approach. Set up the problem again, but this time choose a point Q on line ℓ. Then mark a second point P on ℓ so that $QA = QP$. Draw two circles that pass through point Q: one centered at point A and another centered at point P. These circles should intersect at Q and another point, which we will call B. Mark this point and draw line AB. There are several ways to explain why line AB is parallel to line ℓ. One explanation uses congruent triangles and some basic facts about parallel lines.

Theorem 4. (Parallel Lines) *When a transversal (such as AC below) intersects two parallel lines (AB and DC), alternate interior angles (α and γ) are equal. Conversely, if the alternate interior angles formed by two lines and a transversal are equal, then the two lines are parallel.*

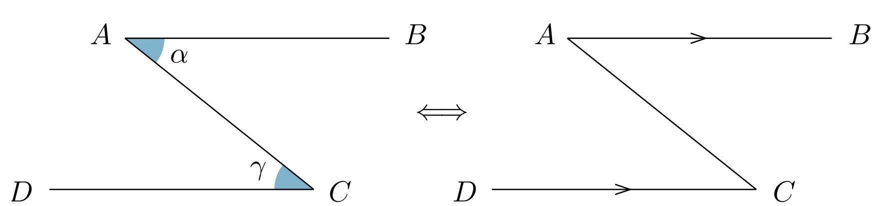

Figure 4: Transversal AC intersects two lines AB and DC.