Impartial Combinatorial Games

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1 Warmups

1.1 (Kozepiskolai Matematikai Lapok, 1980) Contestants B and C play the following game. Player B places a queen on some field of the top row of an $n \times n$ chess board. Then they make moves with the queen alternately according to the rules of chess but they are allowed to move only to fields nearer to the left bottom corner than the field they move from. Player C moves first. The player compelled to move into the left bottom corner loses. Who wins if $n = 8$? Find n such that the other player has a winning strategy.

1.2 (Parabola, 1975) Same as the previous problem, only with a pawn (that is, the piece can only move to an adjacent square, as well as only towards the left bottom forner)

1.3 *A game:* Starting from 1, the players take turns multiplying the current number by any whole number from 2 to 9 (inclusive). The player who first names a number greater than 1000 wins. Which player, if either, can guarantee victory in this game?

1.4 *A game:* There are two piles of candy. One pile contains 20 pieces, and the other 21. Players take turns eating all the candy in one pile and separating the remaining candy into two (not necessarily equal) piles. (A pile may have 0 candies in it) The player who cannot eat a candy on his/her turn loses. Which player, if either, can guarantee victory in this game? *This seems like a NIM variant – see below – but if you don't see an invariant right away, perhaps you can work backwards – you lose if there are no candies left on your turn; you WIN if there is one pile left; From what positions is there nothing you can do to keep your opponent from winning on the next turn? Iterate.*

2 What is a game?

The type of games we are going to analyze, (usually called *Combinatorial Games*) have the following properties:

- There are two players, usually called Left and Right (sometimes Blue and Red), who alternate moves in the game *as a whole*.
- Both players know what is going on, nothing is hidden or private, there is *complete information*, and there is no random element, no dice or shuffled cards, etc.
- A game has several, usually finitely many **positions** and often a designated **starting position**.
- A **move** consists of going from a current position to a new position. The new positions a particular player may choose to go to from a given position are called the **options** for that position. In a given position, Left and Right may have different options.
- in **normal play**, a player unable to move loses.
- The game should always come to an end because some player must eventually be unable to move.

If both players have the same options in every position, the game is called **impartial**, if not, the game is **partizan**.

3 Starter Games

3.1 NIM and variants

Do you know this game? Two players take turns taking stones from a pile. The player who takes the last stone wins. Wait, that sounds too easy . . .

It's pretty simple, but, in fact, there's a sense in which *every* impartial game may be reduced to NIM.

NIM may be a very old game, but the first modern mention of it is a 1901 analysis of the game (actually, there the player who takes the last stone loses, but NIM is more commonly used for the version given here, which has a slightly simpler analysis). You can find many NIM players on the web, and the game itself is exhaustively studied in Martin Gardner (1959), "Mathematical Puzzles and Diversions". And "Winning Ways for your Mathematical Plays" (Berlekamp, Conway, and Guy) devotes several hundred pages to NIM and its variants, using them as the principal models for developing the theory of "surreal numbers"...

The Rules of NIM: A game of NIM starts with several piles of stones. Two players take turns removing stones – on one turn, a player may remove as many stones as he or she likes, but they must all come from the same pile. The player who is able to take the last stone wins.

One common way to play NIM is to start with three piles of stones, one with three stones, one with five, and one with seven. This is not the only way you can play, though.

The invariant that solves NIM may be hard to detect – we can start by working backwards and see if it comes out.

3.5 NIM variant: only one pile of stones, but you can only take $1, 2, 3, \ldots, k$ stones on your turn.

3.6 (Leningrad Math Olympiad, 1988) NIM variant: one-pile NIM, but on your turn you may only take a number of stones that is a power of 2. With perfect play, and a starting positing of 500 stones, which player wins (takes the last stone?)

3.7 NIM variant: only one pile of stones, but you can only take either 1 stone, or a prime number of stones.

3.8 NIM variant: only one pile of stones, but you can only take either 1 stone, or a prime number of stones, or a power of a prime number of stones.

3.9 NIM variant (sort of): two piles, but on one turn you can only take from one pile a positive multiple of the stones in the other pile. Show if one pile is more than twice as big as the other, then the first player can win with perfect play

3.2 Sylver Coinage

Two players take turns naming positive integers that are not the sum of nonnegative multiples of previously named integers. The player who is forced to name 1 loses.

3.3 Dawson's Kayles

3.10 Northcott's Game On a checkerboard, put a blue stone and a red stone in each row. On each player's turn he or she may move his/her color stone in that row, but can't jump over the other player's stone.

3.11 Similar, but one long row with multiple stones on it (all same color). On your turn, you may move any stone to the left without jumping over another stone.

3.12 Dawson's Kayles You have a row of stones. On your turn you can take any two adjacent stones (which may split the row into two rows, unless the two you took were on the extreme left or extreme right). Winner is first player not to move. (In ordinary Kayles you can take either one or two adjacent stones. In some versions, the stones begin in a circle, not a row.)

3.4 non-constructive proofs

Since it is known one player or the other must have a winning strategy (in the absence of draws – see "basic structure theorem" below), if you can show that one player CAN'T have a winning strategy, than the other player MUST – but that doesn't tell you what it is!

This is often done by showing that any winning strategy for a player (typically the one who moves second – but not always) could be mimicked by the other player.

A simple example: TicTacToe – not that you can't come up with a constructive proof, but it is clear that the second player can't guarantee a win without looking at any moves at all.

3.13 The game of CHOMP. (I first heard Paul Halmos describe this game, and it's in one of his books (Problems for Mathematicians Old and Young), but I'm pretty sure it predates them, if not him.) Start with an chocolate bar, ruled into squares ($m \times n$). On a turn, a player may pick any value (i, j) for which the (i, j)th square is as yet uneaten and eat both it and all squares (i', j') for which $i' \geq i$ and $j' \geq j$ (in other words, all squares above and to the right of the given one). The player to eat square $(1, 1)$ (the "poison" square) loses. (you can also do this in more than two dimensions, with fairly obvious modifications of the rules).

First (easyish) show that the first player has a winning strategy in every case except $m = n = 1$. Second (*much* harder, except for some special cases) come up with winning strategies for various boards.

Also,here's a problem that isn't a game:

3.14 Suppose the squares of an 8×8 chessboard are randomly colored black or white (as opposed to the usual alternating pattern). Prove there must be four squares that are the corners of a rectangle (parallel to the sides of the chessboard) that are all the same color. Generalize this: for what $m \times n$ boards could the same be said? (that is, if the board is randomly colored black and white, there must be four squares that form the corner of a rectangle that are all the same color.) A much harder generalization might involve more than 2 colors — for example, is it true that if the squares of a 10×10 board are colored red, white, and blue, there must be four squares that form the corners of a rectangle that are all the same color? What about a 100×100 board?

But compare it to:

3.15 (Kvant, 1983) Two persons play the following game on an infinite square lattice: The first player colors one lattice point red, the second player colors k (as yet uncolored) lattice points blue, the first play colors one more lattice point red, the second k more blue, and so on. The first player's goal is to color four points forming the vertices of a square with sides parallel to the axes of the lattice. Can the second player prevent him from doing this (a) when $k = 1$? (b) when $k = 2$? (c) for some $k > 1$?

3.16 A game begins with a polygon (for the sake of argument, make it a 10 sided polygon, also you can make it convex if you prefer). The players take turns drawing a diagonal of the polygon (provided it does not cross any previously drawn lines). With perfect play who wins? Can you describe a strategy?

actually, this is the same problem as:

3.17 Another game: at the start twenty points are placed on the circumference of a circle. The players take turns connecting any two points by a straight line segment – they are not allowed to cross any line segments already drawn. The player who cannot connect two points in this way loses. Can one player guarantee a win, and if so, how?

4 Various Problems

4.18 Two people play a game by taking turns cutting up a rectangular chocolate bar that is initially 6×8 squares in size. On a turn, the players can only cut a bar along a division between squares and the cut must be in a straight line (and it must sever the piece). Whoever makes the last legal cut wins. Is there a winning strategy for the first or second player? What if the bar is has dimensions $m \times n$?

4.19 (somewhat harder!) Another game: At the start, an $m \times n$ rectangular bar of chocolate, made up of mn unit squares. Each turn, a player can break one piece along a straight division line of the chocolate, forming two pieces. The first player who is able to break off a piece of chocolate that is a single square wins. Can one player guarantee a win, and if so, how? Again, for the sake of argument, we can talk about a 5×10 bar of chocolate.

4.20 Another game: A box has 300 matches. Players take turns removing no more than half the matches in the box. The player who can't move loses. Can one player guarantee a win, and if so, how? Generalize: what if there were n matches in the box?

4.21 Two players play a game: At the start, ten 1s and ten 2s are written on a blackboard. In one turn, a player may erase any two of the numbers. If the two numbers erased were identical, they are replaced by a single 2; if the numbers were different, they are replaced by a single 1. The first player wins if the last number on the board is a 1, the second player wins if a 2 is left. Can one player guarantee a win, and if so, how?

4.22 A game begins with six dots (no three colinear – if you prefer let them form the vertices of a regular hexagon, which is, however, not drawn at the start of the game) The players take turns connecting the dots – each with his or her own color (say Red and Blue). It doesn't matter if lines cross. The first player wins if NO one-color triangles are formed, the second player wins if either player completes a one-color triangle. With perfect play, who wins? (A similar game, but harder to analyze, is that whichever player first makes a triangle of his/her color LOSES).

4.23 Another game: Given an $m \times n$ checkerboard at the start, on each turn a player may cross out all the squares in a row or column that still has at least one square remaining. The player who has no squares left loses. Can one player guarantee a win, and if so, how? The answer may be different for different values of m and n. For the sake of argument, let's use 9×10, 10×12, and 9×11 as sample cases.

4.24 (MATYC journal 1978, also math circles 7.35) *A game:* Starting from 0, the players take turns adding to the current number any whole number from 1 to 9 (inclusive). The winner is the player who reaches 100. Which player, if either, can guarantee victory in this game?

4.25 *A game:* Starting from 2, the players take turns adding to the current number any positive whole number less than the current number. The player who reaches the number 1000 wins. Which player, if either, can guarantee victory in this game?

4.26 (Math circles 17.31, also Leningrad Math Olympiad 1990) *A game:* Starting from the number 1234, the players take turns subtracting from the current number any digit of the number. The player who reaches the number 0 wins. Which player, if either, can guarantee victory in this game?

4.27 (German Math Olympiad, 1984) Given are $2n$ xs ina row. Two players alternately change an x into one of the digits $1, 2, 3, 4, 5$, and 6. The second player wins if and only if the resulting $2n$-digit number (in base ten) is divisible by 9. For which values of n is there a winning strategy for the second player? *Also Vologda Math Olympiad, 1996*

4.28 (Leningrad Math Olympiad, 1987) We are allowed to exchange any two columns or any two rows of an 8×8 chessboard, whose squares (initially) are alternately painted black and white in the usual manner. By a sequence of such exchanges, can we obtain a board whose left half is black and whose right half is white?

4.29 (St. Petersburg Math Olympiad, Spring 1997) In this game, there are 100 coins on one scale and 200 coins on another scale. Each of two players in turn takes one or more coins from one of the scales. The player who makes the number of coins on the two scales equal loses. Scrooge and Glomgold play a game. Scrooge makes the first move. In an errorless game, who wins?

4.30 (St. Petersburg Math Olympiad, Spring 1997) All the natural numbers from 2 to 3500 are written on the blackboard. Players in turn can erase both any non-prime number and all the numbers that are not relatively prime to it (that is, they share a factor). A player wins if, after his turn, only prime numbers remain on the board so that another player can not take a turn. T-Rex and T-Glipp play a game. Who will win if T-Rex begins and neither makes a mistake?

4.31 (USAMO 1999, #5) The Y2K game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O on an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

4.32 (Putnam Competition 1995 B–5) A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking **either**

(a) one bean from a heap, provided that at least two beans are left behind in that heap, **or**

(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

4.33 (Putnam Competition 1993 B–2) Consider the following game, played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

4.34 Two players take turns plugging in whole, non-zero values for the coefficients $a, b, andc$ (not necessarily in that order) in the equation: $x^3 + ax^2 + bx + c = 0$. Prove that no matter what number is used by the second player, the first can always make the three roots of this equation be whole.

4.35 (Journal Rec. Math, 1977) Two players alternate in selecting integers from the set $1, 2, \ldots, n$ until all have been taken. (first player gets the last integer if n is odd). First player wins if either player's total is prime. Otherwise the second player wins. For what n does the first player have the advantage? What if the rules are changed so the *second* player wins when either one is prime?

4.36 (Kozepiskolai Matematikai Lapok, 1984) There are 25 pebbles in a heap. Two players alternately take 1,2, or 3 pebbles until the heap runs out. The winner is the one who takes two for the last time. Which player can win?

4.37 (Journal Rec. Math, 1978) Sulucrus is a NIM variant in which there is only one pile of stones, and one player may remove either 1, 3, or 6 stones on his/her turn and the other player may remove either 2,4 or 5 stones on his/her turn. The winner is the one who takes the last chip. If your opponent allows you to pick a three digit number for the initial number of stones, and choose **either**

- whether you want to go first or second **or**
- whether you wish to be the 1,3,6-stone remover or the 2,4,5-stone remover

(but he/she will make the remaining choice) what conditions should you select?

4.38 (Journal Rec. Math 1981) Nim variant: start with 15 matches. Each player may take either one or two matches per turn. The winner is the player who has taken an odd number of matches when the pile is emptied, regardless of which player took the last match. Which player wins and what is his winning strategy. (Generalize for any odd number of matches)

5 Summary of things we might have done today

We analyzed some simple impartial games by working backwards and classifying positions as belonging to the outcome classes $\mathcal{N}$ (positions where the next player can win) or $\mathcal{P}$ (positions where the previous player can win). e'll add the notation $o(G)$, for the outcome class of a game G.

Theorem ((half of) Fundamental Theorem of Combinatorial Game Theory) *Every* impartial *game meeting the conditions above must belong to either $\mathcal{P}$ or $\mathcal{N}$.*

We introduced the notion of adding games: if G_1 and G_2 are games, the game $G_1 + G_2$ is a game in which, on your turn you pick one of the two games in which you have at least one legal move and make such move in that game only (and then it is your opponent's turn to pick a game and make a move). Note that this means that in the component games, it may well happen that the same player makes multiple consecutive moves).

The Nimsum $\oplus$ of a set of numbers is computed by adding their binary representations "without carrying", (this is the same as bitwise Xor). So for example $9 \oplus 3 = 10$ since 9 is 1001 in base two, and 3 is 11 in base two, and

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ \oplus & & 1 & 1 \end{array}$$

yields 1010 in base two, i.e. 10 in base ten.

We showed that Nim-addition has the following nice properties

Lemma

- *For any two non-negative integers x and y, $x \oplus y = 0$ if and only if $x = y$.*
- *If the nimsum of a set of non-negative integers is 0 and you change one of them at all (let's say from x to y), the nimsum of the set will NOT be 0. (in fact, it will be $x \oplus y$)*
- *If the nimsum of a set of non-negative integers is not 0, then there is a way to decrease at least one of the numbers so that the nimsum is 0. (in fact, if the nimsum of the set is a, we want to change one of the numbers from x to $x \oplus a$ – and we were able to show that there must be at least one integer x in the set for which $x \oplus a$ is smaller than x.*

This gave us the way to determine winning stragies for NIM. We also stated, but did not prove:

Theorem (Sprague-Grundy Theorem) *Every impartial game is equal to a game of NIM with 1 heap of n stones for some value of n.*

What does it mean for two games to be equal, anyway?

Definition *Given two game positions (in* any *combinatorial games) G and H:*

- *$G = H$ means that for every combinatorial game position X, outcome$(G + X) =$ outcome$(H + X)$*

Theorem *All games that are in outcome class $\mathcal{P}$ games are equivalent to each other. If X is any combinatorial game in outcome class $\mathcal{P}$ and Y is any combinatorial game, then $X + Y = Y$.*

Which suggests that it makes sense to assign all games in class $\mathcal{P}$ the number 0.

By adding the notion of the "birthday" of a game, we were able to find the nimber of any impartial game.

5.1 MEX, Dawson's Kayles, octal games, and an open question in combinatorial game theory

We defined the MEX of a set of nimbers to the the "minimal excluded nimber" and that if one has a game position in an impartial game in which the set of game positions you can move to on your turn are all nimbers with MEX $*n$, then this game position is itself the nimber $*n$.

From this and recursion on the birthday of the game, we showed all strictly finite impartial games are equal to a nimber and computed the nimbers for games of Dawson's Kayles up to a size of about 10, after determining that the set of game positions you can move to if you start with a Dawson's Kayle position of consecutive cells of size n are all possible $*k \oplus *(n-2-k)$ as k goes from 0 to $n-2$.

From this, The nim values of Dawson's Kayles may be computed
(see https://oeis.org/A002187):

units digit:	0	1	2	3	4	5	6	7	8	9
0-9	0	0	1	1	2	0	3	1	1	0
10-19	3	3	2	2	4	0	5	2	2	3
20-29	3	0	1	1	3	0	2	1	1	0
30-39	4	5	2	7	4	0	1	1	2	0
40-49	3	1	1	0	3	3	2	2	4	4
50-59	5	5	2	3	3	0	1	1	3	0
60-69	2	1	1	0	4	5	3	7	4	8
70-79	1	1	2	0	3	1	1	0	3	3
80-89	2	2	4	4	5	5	9	3	3	0
90-99	1	1	3	0	2	1	1	0	4	5

The nim values for this game has period 34, with the only exceptions at 1, 15, 17, 18, 32, 35 and 52. After directly computing that the nim numbers repeat on a cycle of 34 up to a bit more than twice the largest exception,

This game is known as an "octal" game. It is an open conjecture in combinatorial game theory whether all finite octal games have periodic nimbers. In particular, "Grundy's game" (Given a pile of n stones, a move is to split it into two unequal heaps.) is not known to be periodic, though its nimbers have been computed for n up to 2^{35}.

We defined a game to be a *number* if, when written as $G = \{G_L|G_R\}$, all of its right options are preferred (by Left) to all of its right options. So not all games are *numbers*.

6 References

The four volume collection "Winning Ways for your Mathematical Plays", by Berlekamp, Conway, and Guy is an amazing tour de force of puns, Combinatorial Game Theory and zillions of games and variants thoroughly analyzed. Some of the math is pretty advanced (this class gave you a head start on it), but large chunks of the books are very readable, too.

"Lessons in Play" by Albert, Nowakowski, and Wolfe, is designed to be a textbook for an undergraduate class on the subject of combinatorial games. It isn't intended to be as funny as Winning Ways, but it's still relatively accessible and does include a lot of specific game analysis.

"Combinatorial Game Theory" by Siegel, is a more abstract approach and is aimed more at graduate students or professional mathematicians who want an introduction or reference to the material. It's a great book, and actually quite readable as high-level math books go, but probably not as accessible for high school students as the others.

"On Numbers and Games" by Conway – is also a high-level book, though written in Conway's playful style. It's also a little out-of-date, published in 1976 (Siegel's book is from 2013 and contains many results obtained in the 35-plus years in between)– but it's still a classic in the field.

There is a serious of books titled "Games of No Chance" (also "More Games of No Chance", "Games of No Chance, Volume 3", etc.) which are collections of short papers on combinatorial game theory resulting from workshops held at MSRI in Berkeley.