Dissections and Decompositions

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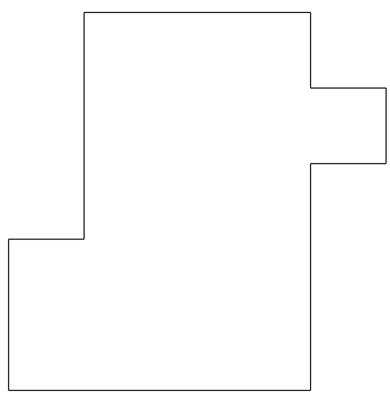
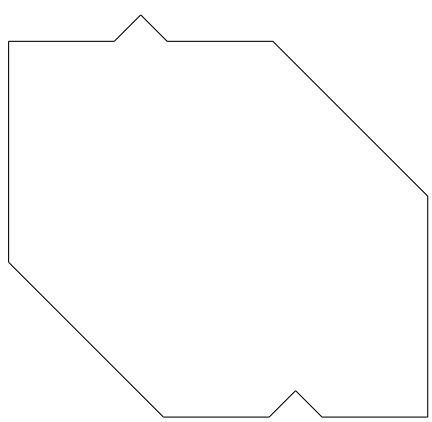
Geometric dissections are ways of cutting up a shape into pieces and in some cases rearranging them to form other shapes. We'll be mostly concerned here with pieces formed by straight line cuts or polygonal pieces. One common type of dissection problem is to try to divide a geometric figure into two or more congruent pieces.

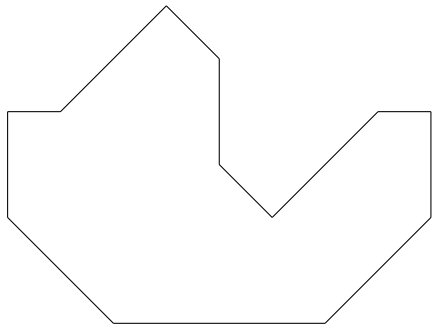
1. Divide the following shapes into two congruent pieces (parts (a) and (b) are from a puzzle by Kimmo Ericsson):

(a)

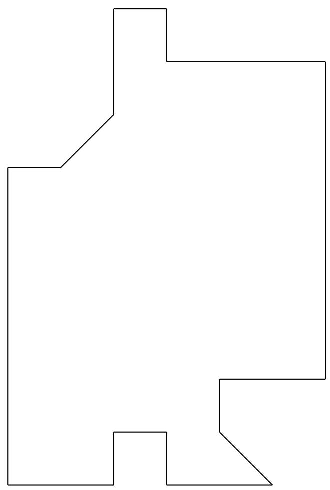


(b)

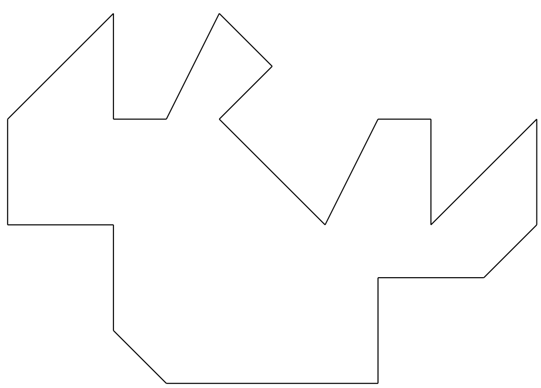




(c)

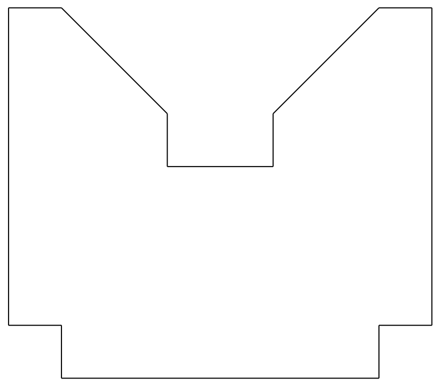


(d)



(e)

2. Divide the following shape into four congruent pieces (puzzle by Nob Yoshighara).



3. When is it possible to divide a shape into two congruent pieces in this manner? What must be true about the boundary of the shape?

Another kind of dissection problem is to try to cut a shape into pieces that can be rearranged to form another shape. If a shape can be cut into pieces which can be rearranged to form another shape, the two are called scissors-congruent. Dissecting one shape into pieces which can be rearranged to form another can be used as a way to prove that two shapes have the same area.

4. Prove the Pythagorean Theorem using dissection.

5. Find the area of a regular dodecahedron with diameter 1.

6. Dissection can also be used to prove algebraic and combinatorial identities, such as:

   (a) $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \frac{1}{2}$

   (b) $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + ... = \frac{1}{3}$

   (c) $1 + 2 + 3 ... + n = \frac{n(n+1)}{2}$

   (d) $1 + 3 + 5 + ... + 2n - 1 = n^2$

   (e) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

   (f) $1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$

   (g) $F_0^2 + F_1^2 + F_2^2 + ... + F_n^2 = F_n \cdot F_{n+1}$

7. The Pizza Theorem states that if a circular pizza is cut by straight lines all passing through one fixed point of the pizza at equal angles, and the total number of pieces is a multiple of 4 greater than or equal to 8, then the sum of the areas of alternate pieces is exactly half the area of the pizza. Prove this using dissection.

We know that if a shape can be dissected to form another shape, the two must have the same area. But what about the other direction? Given that two shapes have the same area, are they necessarily scissors-congruent?

8. Given two polygons with the same area, can one be dissected by straight line cuts into pieces which can be rearranged to form the other? If not, what other conditions are necessary on the polygons for this to be possible? (This problem was solved independently by William Wallace, Farkas Bolyai, and Paul Gerwien.)

9. What happens in three dimensions? Can any polyhedron be cut by planes to form any other polygon? This problem was the third of twenty-three problems proposed by Hilbert in 1900. It was the first of the these problems to be solved, answered in 1901 by Max Dehn.

In decompositions of a shape, we are choosing any subset of the points in this shape (not necessarily connected) and moving these subsets around to form a new shape. In this case, it's a lot harder to see whether the beginning and ending shapes have the same area since the intermediate pieces might not have well defined areas. Two shapes that can be decomposed in this manner are equidecomposable. A paradoxical decomposition is when a shape is equidecomposable with two copies of itself.

10. Do paradoxical decompositions exist in two dimensions? If two shapes are equidecomposable, do they have to have the same area?

11. Conversely, are two shapes with the same area equidecomposable? In particular, is a circle equidecomposable with a square of equal area? (This is Tarsi's Circle-Squaring Problem.)

12. (Banach-Tarski) Do paradoxical decompositions exist in three dimensions?