Berkeley Math Circle: Monthly Contest 7 Solutions

1. Determine, with proof, the value of

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \cdots + 97^2 - 98^2 + 99^2.$$

Solution. Observe that $3^2-2^2 = (3-2)(3+2) = 3+2$, $5^2-4^2 = (5-4)(5+4) = 5+4$, and so on. Thus this sum, call it S, is actually equal to

$$S = 1 + (2+3) + (4+5) \cdots + (97+98) + 99.$$

We can also write it in reverse as

$$S = 99 + 98 + 97 + 96 + 95 + \cdots + 3 + 2 + 1.$$

Adding these two, we get

$$2S = \underbrace{100 + 100 + \cdots + 100}_{99 \text{ times}}.$$

Thus, $S = \frac{1}{2} \cdot 100 \cdot 99 = 4950$. $\square$

2. How many ways are there to color the five vertices of a regular 17-gon either red or blue, such that no two adjacent vertices of the polygon have the same color?

Solution. The answer is zero! Call the polygon $A_1A_2 \ldots A_{17}$. Suppose for contradiction such a coloring did exist.

If we color A_1 red, then A_2 must be blue. From here we find A_3 must be red, then A_4 must be blue; thus A_5 must be red, A_6 must be blue, Proceeding in this manner, we eventually find that A_{15} is red, A_{16} is blue, and then A_{17} is red. But A_1 and A_{17} are adjacent and both red, impossible.

The exact same argument holds if we started by coloring A_1 blue. Therefore, there are no colorings at all with the desired property. $\square$

3. Mr. Fat moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -2y)$, $(-2x, 3y)$, $(x+1, y+4)$ and $(x-1, y-4)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

Solution. Observe that for each of Mr. Fat's moves, the value of $x + y \pmod 5$ is invariant. Therefore, Mr. Fat can never reach $(0, 1)$ from $(0, 0)$. $\square$

4. In convex hexagon $ABCDEF$, $\angle A = \angle B$, $\angle C = \angle D$, and $\angle E = \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point.

Solution. Lines AF, BC, DE determine a triangle Δ whose angle bisectors are the lines in question. Hence the lines concur at the incenter of Δ. $\square$

5. Prove that there exist pairwise distinct positive integers a_0, a_1, a_2, $\ldots$, a_{1000} such that
$$a_0! = a_1!a_2!\ldots a_{1000}!.$$
Here $n! = 1 \times 2 \times \cdots \times n$ as usual.

Solution. We proceed by induction on $n \geq 2$. First, we can have $a_1 = 3$, $a_2 = 5$ and $a_0 = 6$. Now, given a working tuple $(a_0, a_1, \ldots, a_n)$, note that the tuple
$$(a_0!, a_1, \ldots, a_n, (a_0 - 1)!)$$
is a working tuple of length $n + 1$. This completes the proof. □

6. Let positive reals a, b, c obey $a + b + c = 1$. Prove that
$$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \leq \sqrt{3}.$$

Solution. By the Cauchy-Schwarz inequality, we have
$$\begin{aligned} 3 &= \left(\left(a + \frac{(b-c)^2}{4}\right) + \left(b + c - \frac{(b-c)^2}{4}\right)\right)(1+2) \\ &\geq \left(\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{2(b+c) - \frac{(b-c)^2}{2}}\right)^2. \end{aligned}$$
Thus, it suffices to prove that
$$\sqrt{2(b+c) - \frac{(b-c)^2}{2}} \geq \sqrt{b} + \sqrt{c}.$$
If we square both sides, this is equivalent to the assertion that
$$(\sqrt{b} - \sqrt{c})^2 \geq \frac{(b-c)^2}{2} \iff 2 \geq (\sqrt{b} + \sqrt{c})^2$$
which follows by simply observing $b + c \leq 1$. □

7. Let ABC be an acute triangle with circumcenter O and incenter I. Points E, M lie on AC and F, N on AB so that $BE \perp AC$, $CF \perp AB$, $\angle ABM = \angle CBM$ and $\angle ACN = \angle BCN$. Prove that I lies on EF if and only if O lies on MN.

Solution. Let $a = BC$, $b = CA$, $c = AB$. It is well-known (and follows from, say, Stewart's Theorem) that $AM = \frac{bc}{a+c}$ and $AN = \frac{bc}{a+b}$.
Now, the distances from O to BC, AC, AB are $R\cos\alpha$, $R\cos\beta$, $R\cos\gamma$, respectively, where α, β, γ are the angles of $\triangle ABC$, and R is the circumradius of ABC.
So, O is on line MN if and only if
$$\begin{aligned} &[ANM] = [ANO] + [AOM] \\ \iff & [ABC] \cdot \frac{AN}{AB} \cdot \frac{AM}{AC} = [ANO] + [AOM] \iff \left(\frac{1}{2}aR\cos\alpha + \frac{1}{2}bR\cos\beta + \frac{1}{2}cR\cos\gamma\right) \cdot \frac{b}{a+b} \cdot \end{aligned}$$

(Here we use $[\mathcal{P}]$ for the area of polygon $\mathcal{P}$.) Next, recall that $AE = c\cos\alpha$, $AF = b\cos\alpha$. Thus I is on EF if and only if

$$\begin{aligned} [AFE] &= [AFI] + [AIE] \\ \iff [ABC] \cdot \frac{AF}{AB} \cdot \frac{AE}{AC} &= \frac{1}{2} r \cdot c\cos\alpha + \frac{1}{2} r \cdot b\cos\alpha \\ \iff \frac{1}{2} r(a+b+c)\cos^2\alpha &= \frac{1}{2} r(b+c)\cos\alpha \\ \iff (a+b+c)\cos\alpha &= b+c. \end{aligned}$$

Because $AC = AE+EC$, we know $b = c\cos\alpha + a\cos\gamma$. Similarly, $c = b\cos\alpha + a\cos\beta$. Thus I is on EF if and only if

$$\begin{aligned} (a+b+c)\cos\alpha &= (c\cos\alpha + a\cos\gamma) + b(\cos\alpha + a\cos\beta) \\ \iff \cos\alpha &= \cos\beta + \cos\gamma. \end{aligned}$$

This implies the result. $\square$