Berkeley Math Circle: Monthly Contest 7

Due April 5, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 5, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 7

1. Determine, with proof, the value of
$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \cdots + 97^2 - 98^2 + 99^2.$$

2. How many ways are there to color the five vertices of a regular 17-gon either red or blue, such that no two adjacent vertices of the polygon have the same color?

3. Mr. Fat moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -2y)$, $(-2x, 3y)$, $(x+1, y+4)$ and $(x-1, y-4)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

4. In convex hexagon $ABCDEF$, $\angle A = \angle B$, $\angle C = \angle D$, and $\angle E = \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point.

5. Prove that there exist pairwise distinct positive integers a_0, a_1, a_2, $\ldots$, a_{1000} such that
$$a_0! = a_1! a_2! \ldots a_{1000}!.$$
Here $n! = 1 \times 2 \times \cdots \times n$ as usual.

6. Let positive reals a, b, c obey $a+b+c=1$. Prove that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}\le\sqrt{3}.$$

7. Let ABC be an acute triangle with circumcenter O and incenter I. Points E, M lie on AC and F, N on AB so that $BE\perp AC$, $CF\perp AB$, $\angle ABM=\angle CBM$ and $\angle ACN=\angle BCN$. Prove that I lies on EF if and only if O lies on MN.