Berkeley Math Circle: Monthly Contest 5

Due February 2, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 2, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 5

1. Let A, B, C be points in a line on this order. A car travels from A to B at 40 kilometers per hour and then from B to C at 60 kilometers per hour, without stopping in between. What is the overall speed of the trip from A to C?

2. For positive integers n, prove that $\gcd(6n+1, 15n+2) = 1$.

3. A line in the Cartesian plane is called *stable* if it passes through at least two points (x, y) such that x and y are rational numbers. Prove or disprove: every point lies on some stable line.

4. There are three boxes of stones. Each hour, Sisyphus moves a stones from one box to another. For each transfer of a stone, he receives from Zeus a number of coins equal to the number of stones in the pile from which the stone is drawn minus the number of stones in the recipient pile, with the stone Sisyphus just carried not counted. If this number is negative, Sisyphus pays the corresponding amount (and can pay later if he is broke).

   After 1000 years, all the stones lie in their initial boxes. What is the greatest possible earning of Sisyphus at that moment?

5. Let a, b, c be positive reals. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

6. Let $\mathcal{P}$ be a regular 17-gon; we draw in the $\binom{17}{2}$ diagonals and sides of $\mathcal{P}$ and paint each side or diagonal one of eight different colors. Suppose that there is no triangle (with vertices among vertices of $\mathcal{P}$) whose three edges all have the same color. What is the maximum possible number of triangles, all of whose edges have *different* colors?

7. Let ABC be an acute triangle with orthocenter H and altitudes BD, CE. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.