Berkeley Math Circle: Monthly Contest 4 Solutions

1. A thief starts at the point $x = 0$ on a number line, and runs either left or right at a constant rate. One hour later, a policeman who moves twice as fast as the thief appears at the point $x = 0$. However, the policeman does not know which direction the thief went and cannot see the thief. Show that the policeman can still catch the thief.

   Solution. Assume for simplicity that the rate at which the thief moves is 1 unit per hour. The police should drive left for an hour, then drive right for four hours.

   This works for the following reason: if the thief runs left, he will be caught in one hour by the left-moving police and the position $x = -2$. If the thief runs right, then after the first hour of the police's movement, the thief will be at the position $x = +2$ while the officer is at position $x = -2$. Then, by going right for four hours, the police will catch the robber at the position $x = +6$. $\square$

2. Determine, with proof, the value of $\log_2 3 \log_3 4 \log_4 5 \ldots \log_{255} 256$.

   Solution. We use the fact that $\log_b a = \frac{\log a}{\log b}$. Thus, the product equals

$$\frac{\log 3}{\log 2}\frac{\log 4}{\log 3}\cdots\frac{\log 256}{\log 255} = \frac{\log 256}{\log 2} = \log_2(256) = 8.$$

   $\square$

3. Consider the four points $A = (2, 4)$, $B = (0, 0)$, $C = (6, 0)$, and $D = (6, 3)$ in the plane. For which point P in the plane is the sum $PA + PB + PC + PD$ minimized?

   Solution. The answer is $P = (4, 2)$, which is the intersection of the line segments AC and BD. To see this is optimal, note that by the triangle inequality we have

$$PA + PC \geq AC \quad \text{and} \quad PB + PD \geq BD$$

   so $PA + PB + CD + PD \geq AC + BD$ with equality occurring for only the point P we named earlier. $\square$

4. Let a, b, c be positive integers such that $a^2 - bc$ is a perfect square. Prove that the number $2a + b + c$ is not a prime number.

   Solution. Suppose that $a^2 - bc = d^2$, so that $(a - d)(a + d) = bc$. Then, we can find positive integers w, x, y, z such that $a - d = wx$, $a + d = yz$, $b = wy$, $c = xz$. Thus, $2a + b + c = (a - d) + (a + d) + b + c = wx + wy + xz + yz = (w + z)(x + y)$ which is clearly not prime. $\square$

5. For positive integers x_1, x_2, …, x_n satisfying $x_1 + \cdots + x_n = 101n$, prove that

$$\binom{x_1}{2} + \binom{x_2}{2} + \cdots + \binom{x_n}{2} \ge 5050n.$$

Solution. Noting that $22\binom{x}{2} = x^2 - x$, we see that it suffices to prove that

$$x_1^2 + \cdots + x_n^2 \ge 2(5050n) + 101n = 10201n.$$

This follows immediately by Cauchy-Schwarz as

$$(101n)^2 = (x_1 + \cdots + x_n)^2 \le (x_1^2 + \cdots + x_n^2)(1 + \cdots + 1) = n(x_1^2 + \cdots + x_n^2).$$

Rearranging gives the conclusion. □

6. In triangle ABC, a point M is selected in its interior so that $\angle MAB = 10°$, $\angle MBA = 20°$, $\angle MCA = 30°$ and $\angle MAC = 40°$. Determine the value of $\angle MBC$.

Solution. Let X be on side BC so that AM bisects $\angle ABX$. Let Y be on side AC so that BM bisects $\angle ABY$. Denote by Z the intersection of these two lines; thus M is the incenter of $\triangle ABZ$. Then, $\angle BMZ = 90°dg + \frac{1}{2}\angle BAZ = 100°$. On the other hand, $\angle BMC = 180° - (\angle B - 20°) - (\angle C - 30°) = 100°$ as well, so we conclude that M, Z, C are collinear. Moreover, from $\angle AMB = 150°$ we obtain $\angle AZB = 120°$, and hence the lines AX, BY, CM not only concur at Z but form $60°$ angles to each other.

Now, $\angle ZAC = \angle ZCA = 30°$. From this we deduce $ZA = ZC$, and hence line BZ is the perpendicular bisector of side AC. Thus triangle ABC is isosceles, with $AB = BC$. As $\angle A = 50°$, we get $\angle B = 80°$, and thus $\angle MBC = 60°$. □

7. A thief starts at the point $x = 0$ on a number line, and runs either left or right at a constant rate. Many hours later, a policeman who moves twice as fast as the thief appears at the point $x = 0$. However, the policeman does not know which direction the thief went and cannot see the thief; moreover, the policeman does not know how long ago the thief departed. Prove that the policeman can still catch the thief (in a finite amount of time).

Solution. Assume for simplicity that the rate at which the thief moves is 1 unit per hour. For convenience, we say the police is starting at time $t = 1$ at position $x = 0$. We claim that the following strategy suffices: from the time $t = 4^k$ to $t = 4^{k+1}$, drive left if k is even and right if k is odd.

Let x_k denote the policeman's time at time $t = 4^k$. Then $x_0 = 0$, and thereafter,

$$x_k = \left(4^k - 4^{k-1}\right) \cdot 2 \cdot (-1)^k + x_{k-1}.$$

The first few terms of the sequence are $x_1 = -6$, $x_2 = 18$, $x_3 = -78$, $x_4 = 305$. By induction, it is not hard to find the general formula

$$x_k = \frac{6}{5}\left((-4)^k - 1\right).$$

Note that $x_k \geq 0$ exactly when k is even.

Now, if the thief runs right starting at time $t = -a$, then there is a sufficiently large even k such that $x_k/4^k > 4^k + a$, which implies the thief will be caught in this case. The same is true if the thief runs left. □