Berkeley Math Circle: Monthly Contest 3

Due December 1, 2015

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 3 is due on December 1, 2015.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 3

1. Suppose x and y are real numbers satisfying $x + y = 5$. What is the largest possible value of $x^2 + 2xy$?
2. Let ABC be an acute triangle with orthocenter H, circumcenter O, and incenter I. Prove that ray AI bisects $\angle HAO$.
3. For which prime numbers p is $p^2 + 2$ also prime? Prove your answer.
4. There is a colony consisting of 100 cells. Every minute, a cell dies with probability $\frac{1}{3}$; otherwise it splits into two identical copies. What is the probability that the colony never goes extinct?
5. Let H, I, O, Ω denote the orthocenter, incenter, circumcenter and circumcircle of a scalene triangle ABC. Prove that if $\angle BAC = 60°$ then the circumcenter of $\triangle IHO$ lies on Ω.
6. Let a, b, c be positive integers. Prove that it is not possible for $a^2 + b + c$, $b^2 + c + a$, $c^2 + a + b$ to all be perfect squares.

7. Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.