Berkeley Math Circle: Monthly Contest 1 Solutions

1. Let $ABCDE$ be a convex pentagon with perimeter 1. Prove that triangle ACE has perimeter less than 1.

   Solution. By the triangle inequality, we have $AC < AB + BC$ and $CE < CD + DE$, so $AC + CE + EA < AB + BC + CD + DE + EA$ as desired. □

2. Show that there are infinitely many prime numbers whose last digit is not 1.

   Solution. Assume there are only finitely many such primes $p_1, \ldots, p_k$. Consider the number
   $$N = 10p_1p_2 \ldots p_k - 1.$$
   Since N has last digit 9, there must be a prime p dividing N which does not have last digit 1 (otherwise N must have last digit 1). But by construction, p cannot divide N, because it leaves a remainder of $p - 1$ when divided by p. This is a contradiction, so our assumption was wrong and there must be infinitely many such primes. □

3. Let $P(x)$ be a nonzero polynomial with real coefficients such that
   $$P(x) = P(0) + P(1)x + P(2)x^2$$
   holds for all x. What are the roots of $P(x)$?

   Solution. Let $c = P(0)$. Selecting $x = 1$ in the given, we have that $P(0) + P(1) + P(2) = P(1)$, so $P(2) = -c$. Selecting $x = 2$, we find that
   $$-c = c + 2P(1) + 4(-c) \implies P(1) = c.$$
   Therefore, $P(x) = c + cx - cx^2$. As $c \neq 0$, we solve $1 + x - x^2 = 0$ to get the answers $x = \frac{1}{2}(1 \pm \sqrt{5})$. □

4. There are seven green amoeba and three blue amoeba in a dish. Every minute, each amoeba splits into two identical copies; then, we randomly remove half the amoeba (thus there are always 10 amoeba remaining). This process continues until all amoeba are the same color. What is the probability that this color is green?

   Solution. The answer is 70%. Rather than considering the given colors, imagine the amoeba are colored with the following ten colors:

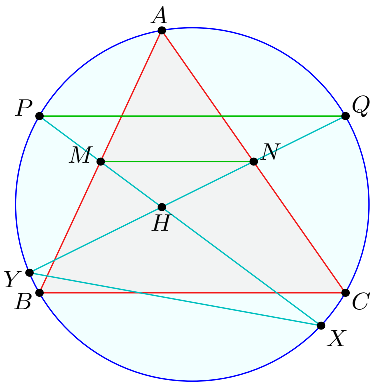
   - Forest green
   - Hunter green
   - Jungle green
   - Myrtle green
   - Mint green

- Tea green
- Aqua green
- Sky blue
- Navy blue
- Midnight blue

Now, we simply wait until one of these particular shades becomes the only remaining shade. By symmetry, each shade has a 10% chance of being the sole survivor. Thus, the chance it is a "green" shade is 70%. □

5. Let ABC be an acute triangle with orthocenter H, and let M and N denote the midpoints of AB and AC. Rays MH and NH intersect the circumcircle of ABC again at points X and Y. Prove that the four points M, N, X, Y lie on a circle.

   Solution. Let Ω be the circumcircle of $\triangle ABC$. Let P and Q be the reflections of H across M and N.



   Since $\angle APB = \angle AHB = 180^\circ - \angle C$, point P lies on Ω; thus so does Q. Moreover, $MN \parallel PQ$, so
   $$\angle XMN = \angle XPQ = \angle XYQ$$
   and we are done. □

6. We take a 6×6 chessboard, which has six rows and columns, and indicate its squares by (i, j) for $1 \le i, j \le 6$. The kth *northeast diagonal* consists of the six squares satisfying $i - j \equiv k \pmod 6$ (and so there are six such diagonals); hence there are six such diagonals.

   Determine if it is possible to fill the entire chessboard with the numbers $1, 2, \dots, 36$ (each exactly once) such that each row, each column, and each of the six northeast diagonals has the same sum.

   Solution. The answer is no. Assume for contradiction such a coloring existed; then each row, column and northeast diagonal would have sum exactly
   $$N = \frac{1}{6}(1 + 2 + \cdots + 36) = 111.$$

Now consider the marked squares shown below.

$$\begin{array}{cccccc} A & B & A & B & A & B \\ & C & & C & & C \\ A & B & A & B & A & B \\ & C & & C & & C \\ A & B & A & B & A & B \\ & C & & C & & C \end{array}$$

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ denote the sum of numbers of the squares labelled A, B, C, respectively. We deduce that

$$\begin{aligned} \mathcal{A} + \mathcal{B} &= 3N \\ \mathcal{B} + \mathcal{C} &= 3N \\ \mathcal{A} + \mathcal{C} &= 3N \end{aligned}$$

by considering three rows, three columns, and three northeast diagonals. Summing all these equations gives

$$2\left(\mathcal{A} + \mathcal{B} + \mathcal{C}\right) = 9N$$

which is impossible, because the left-hand side is even while the right-hand side is odd. □

7. Let $a_1, \ldots, a_n$ be distinct integers. Prove that the polynomial

$$(x - a_1)(x - a_2)\ldots(x - a_n) - 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients (i.e. it is irreducible over the integers).

Solution. Assume there exist polynomials f and g satisfying

$$f(x)g(x) = (x - a_1)\ldots(x - a_n) - 1.$$

Let $h(x) = f(x) + g(x)$. Now, for every a_i we have $f(a_i)g(a_i) = -1$, so

$$h(a_i) = f(a_i) + g(a_i) = 0.$$

Thus $h(x)$ has at least n distinct roots, so it has degree at least n. But since $\deg f + \deg g = n$, this can only occur if one of f and g is a constant polynomial, which is what we wanted to prove. □