Berkeley Math Circle: Monthly Contest 1

Due October 6, 2015

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on October 6, 2015.

- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

    BMC Monthly Contest 1, Problem 3
    Bart Simpson
    Grade 5, BMC Beginner
    from Springfield Middle School, Springfield

    Submit **different problems on different pages** as they are graded separately.

- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 1

1. Let $ABCDE$ be a convex pentagon with perimeter 1. Prove that triangle ACE has perimeter less than 1.

2. Show that there are infinitely many prime numbers whose last digit is not 1.

3. Let $P(x)$ be a nonzero polynomial with real coefficients such that
$$P(x) = P(0) + P(1)x + P(2)x^2$$
holds for all x. What are the roots of $P(x)$?

4. There are seven green amoeba and three blue amoeba in a dish. Every minute, each amoeba splits into two identical copies; then, we randomly remove half the amoeba (thus there are always 10 amoeba remaining). This process continues until all amoeba are the same color. What is the probability that this color is green?

5. Let ABC be an acute triangle with orthocenter H, and let M and N denote the midpoints of AB and AC. Rays MH and NH intersect the circumcircle of ABC again at points X and Y. Prove that the four points M, N, X, Y lie on a circle.

6. We take a 6×6 chessboard, which has six rows and columns, and indicate its squares by ordered pairs (i, j) for $1 \leq i \leq j \leq 6$. The kth *northwest diagonal* consists of the six squares (i, j) satisfying $i - j \equiv k \pmod 6$; hence there are six such diagonals. For example, the 4th northwest diagonal is shown in bold below.

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$\mathbf{(5,1)}$	$(6,1)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$\mathbf{(6,2)}$
$\mathbf{(1,3)}$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,4)$	$\mathbf{(2,4)}$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,5)$	$(2,5)$	$\mathbf{(3,5)}$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,6)$	$(2,6)$	$(3,6)$	$\mathbf{(4,6)}$	$(5,6)$	$(6,6)$

Determine if it is possible to fill the entire chessboard with the numbers $1, 2, \ldots, 36$ (each exactly once) such that each row, each column, and each of the six northwest diagonals has the same sum.

7. Let a_1, $\ldots$, a_n be distinct integers. Prove that the polynomial

$$(x - a_1)(x - a_2) \ldots (x - a_n) - 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients (i.e. it is irreducible over the integers).