BAMO
Bay Area Mathematical Olympiad

18th Bay Area Mathematical Olympiad

BAMO-8 Exam

February 23, 2016

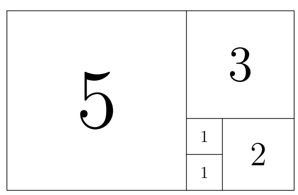
The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems A and B are on this side; problems C, D, E on the other side.

A The diagram below is an example of a *rectangle tiled by squares*:



Each square has been labeled with its side length. The squares fill the rectangle without overlapping.

In a similar way, a rectangle can be tiled by nine squares whose side lengths are 2, 5, 7, 9, 16, 25, 28, 33, and 36. Sketch one such possible arrangement of those squares. They must fill the rectangle without overlapping. Label each square in your sketch by its side length, as in the picture above.

B A weird calculator has a numerical display and only two buttons, $\boxed{\text{D}\sharp}$ and $\boxed{\text{D}\flat}$. The first button doubles the displayed number and then adds 1. The second button doubles the displayed number and then subtracts 1. For example, if the display is showing 5, then pressing the $\boxed{\text{D}\sharp}$ produces 11. If the display shows 5 and we press $\boxed{\text{D}\flat}$, we get 9. If the display shows 5 and we press the sequence $\boxed{\text{D}\sharp}$, $\boxed{\text{D}\flat}$, $\boxed{\text{D}\sharp}$, $\boxed{\text{D}\sharp}$, we get a display of 87.

(i) Suppose the initial displayed number is 1. Give a sequence of exactly eight button presses that will result in a display of 313.

(ii) Suppose the initial displayed number is 1, and we then perform exactly eight button presses. Describe all the numbers that can possibly result? Prove your answer by explaining how all of these numbers can be produced, and that no other numbers can be produced.

C The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5.

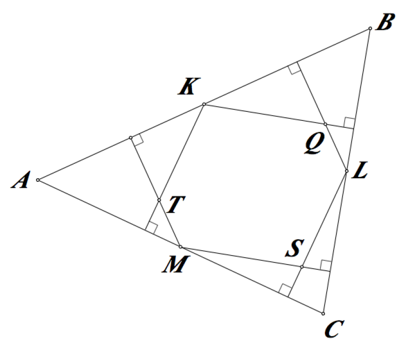
Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$).

Show that, for every integer k greater than or equal to 2,

(i) A and B have the same set of distinct prime factors.

(ii) $A+1$ and $B+1$ have the same set of distinct prime factors.

D In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc. The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon $KQLSMT$ inside triangle ABC. Prove that the area of this hexagon $KQLSMT$ is half of the area of the original triangle ABC.



E For $n > 1$, consider an $n \times n$ chessboard and place identical pieces at the centers of different squares.

(i) Show that no matter how $2n$ identical pieces are placed on the board, that one can always find 4 pieces among them that are the vertices of a parallelogram.

(ii) Show that there is a way to place $(2n-1)$ identical chess pieces so that no 4 of them are the vertices of a parallelogram.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2016 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 2–4PM on Sunday, March 20 (note that this is a week later than last year). This event will include a mathematical talk by **Jacob Fox (Stanford University)**, refreshments, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor and/or `bamo.org` for a more detailed schedule, plus directions.

You may freely disseminate this exam, but please do attribute its source (Bay Area Mathematical Olympiad, 2016, created by the BAMO organizing committee, bamo@msri.org). For more information about the awards ceremony, or with any other questions about BAMO, please contact Paul Zeitz at zeitzp@usfca.edu.