Frieze patterns

1. In the following table of numbers find a pattern that connects adjacent numbers and allows to extend the table indefinitely to the right and to the left.

	0		0		0		0		0		0		0		0		0		0		0		0	
1		1		1		1		1		1		1		1		1		1		1		1		1
	1		2		2		3		1		2		4		1		2		2		3		1	
3		1		3		5		2		1		7		3		1		3		5		2		1
	2		1		7		3		1		3		5		2		1		7		3		1	
3		1		2		4		1		2		2		3		1		2		4		1		2
	1		1		1		1		1		1		1		1		1		1		1		1	
0		0		0		0		0		0		0		0		0		0		0		0		0

Hint: Look at the rhombi

	b	
a		d
	c	

Write the rule that you have just found in the box below:

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \tag{1}$$

Definition. A *frieze* is a grid of numbers bounded from above by an infinite row of 0's, followed by a row of 1's and satisfying the *frieze rule* (1). A frieze is called *closed* if it is also bounded from below by a line of 1's (followed by a line of 0's). The number of nontrivial lines in a closed frieze is called the *width* of the frieze. A frieze is called *integral* if it consists of integers.

2. Prove that in the frieze

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & & \dots & \\
 & & a_1 & & a_2 & & a_3 & & a_4 & & \\
 & & & f_2 & & g_2 & & h_2 & & \dots & \\
 & & & & f_3 & & g_3 & & h_3 & & \\
 & & & & & \ddots & & \ddots & & \ddots & \\
 & & & & & & f_{n-3} & & g_{n-3} & & h_{n-3} \\
 & & & & & & & 1 & & 1 & \\
 & & & & & & & & 0 & &
\end{array}$$

with $f_0 = 1$ and $f_1 = a_1$ the following relation holds

$$f_i = a_i f_{i-1} - f_{i-2}. \tag{2}$$

Hint. Check the statement for $i = 2$ and then use the row of g_i's and induction on i.

3. Use relations (1) and (2) to show that if all the numbers a_i are positive integers than the frieze is integral and all it's entries are positive.

4. Prove that every line of a closed frieze of width $n - 3 \geqslant 1$ is n-periodic.

Hint. Plug $i = n - 1$ into relation (2) and then consider a diagonal starting with f_{n-3} and going North-East.

5. Prove that a closed integral frieze with the first nontrivial line (a_i) has $a_j = 1$ for some j.

Hint. Assume that there is no such j. Show that $f_i > f_{i-1}$ by induction on i.

6. Consider a closed integral frieze F of width w and choose j such that $a_j = 1$. By problem 4, the frieze F is periodic with period $n = w + 3$. Let F' be the $n - 1$-periodic frieze with the first nontrivial line

$$(\dots, a_{j-1}, a_j, a_{j+1}, a_{j+2}, \dots) \longrightarrow (\dots, a_{j-2}, a_{j-1} - 1, a_{j+1} - 1, a_{j+2}, \dots)$$

Prove that the frieze F' is a closed integral frieze of width $w - 1$. Note that you can construct the frieze F from the frieze F' by reverting the process.

Hint. Consider the diagonal f'_i for the frieze F' and show that $f'_i = f_i$ if $i \leqslant j - 2$ and $f'_i = f_{i+1}$ if $i \geqslant j - 1$.

7. Consider a triangulated n-gon. For every vertex v_i of the n-gon, let a_i be its number of adjacent triangles; this yields an n-periodic sequence (a_i). Taking this sequence as the first nontrivial line, we define the frieze corresponding to the triangulation. Draw the n-gon and its triangulation corresponding to the frieze in problem 1.

8. Prove the following theorem by Conway and Coxeter.

Theorem.

(i) The frieze corresponding to a triangulation of an n-gon is a closed integral frieze of width $n - 3$.

(ii) Every closed integral frieze of width $n - 3$ corresponds to some triangulation of an n-gon.

Hint. Consider $n = 3$ and then prove the theorem using induction on n.

The presentation of material in this worksheet completely follows

Claire-Soizic Henry, *Coxeter Friezes and Triangulations of Polygons*, Amer. Math. Monthly **120**, 2013, 553-558.