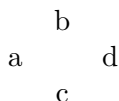


Frieze patterns

1. In the following table of numbers find a pattern that connects adjacent numbers and allows to extend the table indefinitely to the right and to the left.

	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	2	3	1	2	4	1	2	2	3	1	3	1		
3	1	3	5	2	1	7	3	1	3	5	2	1	7	3	1	1
	2	1	7	3	1	3	5	2	1	7	3	1				
3	1	2	4	1	2	2	3	1	2	4	1	2	4	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Hint: Look at the rhombi



Write the rule that you have just found in the box below:

(1)

Definition. A *frieze* is a grid of numbers bounded from above by an infinite row of 0's, followed by a row of 1's and satisfying the *frieze rule* (1). A frieze is called *closed* if it is also bounded from below by a line of 1's (followed by a line of 0's). The number of nontrivial lines in a closed frieze is called the *width* of the frieze. A frieze is called *integral* if it consists of integers.

2. Prove that in the frieze

	0	0	0	0	0					
	1	1	1	1	1	...				
		a_1	a_2	a_3	a_4	...				
			f_2	g_2	h_2	...				
				f_3	g_3	h_3	...			
							
					f_{n-3}	g_{n-3}	h_{n-3}			
						1	1			
							0			

with $f_0 = 1$ and $f_1 = a_1$ the following relation holds

$$f_i = a_i f_{i-1} - f_{i-2}. \quad (2)$$

Hint. Check the statement for $i = 2$ and then use the row of g_i 's and induction on i .

3. Use relations (1) and (2) to show that if all the numbers a_i are positive integers than the frieze is integral and all it's entries are positive.

4. Prove that every line of a closed frieze of width $n - 3 \geq 1$ is n -periodic.

Hint. Plug $i = n - 1$ into relation (2) and then consider a diagonal starting with f_{n-3} and going North-East.

5. Prove that a closed integral frieze with the first nontrivial line (a_i) has $a_j = 1$ for some j .

Hint. Assume that there is no such j . Show that $f_i > f_{i-1}$ by induction on i .

6. Consider a closed integral frieze F of width w and choose j such that $a_j = 1$. By problem 4, the frieze F is periodic with period $n = w + 3$. Let F' be the $n - 1$ -periodic frieze with the first nontrivial line

$$(\dots, a_{j-1}, a_j, a_{j+1}, a_{j+2}, \dots) \longrightarrow (\dots, a_{j-2}, a_{j-1} - 1, a_{j+1} - 1, a_{j+2}, \dots)$$

Prove that the frieze F' is a closed integral frieze of width $w - 1$. Note that you can construct the frieze F from the frieze F' by reverting the process.

Hint. Consider the diagonal f'_i for the frieze F' and show that $f'_i = f_i$ if $i \leq j - 2$ and $f'_i = f_{i+1}$ if $i \geq j - 1$.

7. Consider a triangulated n -gon. For every vertex v_i of the n -gon, let a_i be its number of adjacent triangles; this yields an n -periodic sequence (a_i) . Taking this sequence as the first nontrivial line, we define the frieze corresponding to the triangulation. Draw the n -gon and its triangulation corresponding to the frieze in problem 1.

8. Prove the following theorem by Conway and Coxeter.

Theorem.

(i) *The frieze corresponding to a triangulation of an n -gon is a closed integral frieze of width $n - 3$.*

(ii) *Every closed integral frieze of width $n - 3$ corresponds to some triangulation of an n -gon.*

Hint. Consider $n = 3$ and then prove the theorem using induction on n .

The presentation of material in this worksheet completely follows

Claire-Soizic Henry, *Coxeter Friezes and Triangulations of Polygons*, Amer. Math. Monthly **120**, 2013, 553-558.