Combinatorial Games II

Berkeley Math Circle, Intermediate Level I

April 21, 2015

In this session we will continue playing some *combinatorial games*. After the warm-up of last week, this week we will try to analyze some games more systematically. We will start by analyzing small games of *Chomp*.

1. **Chomp.** This is a game that was invented in the 1970's by David Gale, who was a professor of mathematics here at the University of California, Berkeley.

   Chomp is played on a rectangular chocolate bar divided into smaller squares. Unfortunately, the bottom left square is poisoned. The players take turns, and at each step a player chooses an uneaten chocolate square and eats everything above and to the right of this square. The person who eats the poisoned square loses.

   Who wins the game for a 2×2 square? How do you prove this? How do you analyze the game? We will discuss this example in detail.

   Using what we just discussed about analyzing Chomp, can you determine who wins when starting from a 2×3 rectangle? How about a 3×3 square?

   A nice problem to think about: can you determine which player has a winning strategy in general, given a rectangular chocolate bar to start with? *Hint:* you do not need to know the winning strategy to figure this out!

With our knowledge of how to analyze games like this in general, we are now ready to tackle *Nim*. Last week we understood who wins the game when there are only two piles. Now we will tackle the case of three piles first, and then later any number of piles!

2. **Nim.** In this game there are several piles, each containing finitely many chips. Two players alternate taking turns, and in each turn a player can remove any positive number of chips from a single pile. The winner is the player who removes the last chip.

   Last week we understood the case of two piles. If the number of chips in the two piles are not equal, then the first player has a winning strategy, while if they are equal, then the second player has a winning strategy. We also determined these winning strategies.

   Now let's understand the game with three piles. Play with your neighbor! Try several initial conditions with small number of initial chip counts: $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$, $(2, 2, 3)$, or $(2, 3, 5)$. After playing a few games with your neighbor, try analyzing who wins in these cases! Feel free to analyze together with your neighbor.

   Do you notice any pattern? What is common in the **N** positions? What is common in the **P** positions? (We will discuss what **N** and **P** positions are at the beginning of the session.) *Hint:* Write the number of chips in each pile in binary, and write them down on top of each other. Do you see a pattern now? Can you determine the winning strategy?