Combinatorics and Recurrence 2: Catalan Numbers
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Warm-Up

If n and k are nonnegative integers, "n choose k" is the number of ways of choosing k objects from a set of n objects. This is written $\binom{n}{k}$.

The formula is $\frac{n!}{k!\cdot(n-k)!}$. The exclamation point means factorial. $n!$ equals the product $1 \cdot 2 \cdot \ldots \cdot n$. Also, 0! is defined to be 1.

1. Complete the list:

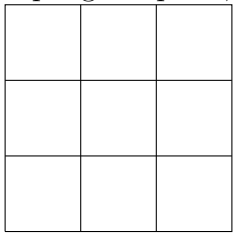
$0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 1$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! =$ _____
$5! =$ _____

2.

(a) How many ways are there to choose three objects from a group of 5?

(b) How many ways are there to choose four objects from a group of 8?

3. In the grid below, how many ways are there to travel from the bottom-left square to the top-right square, if you're only allowed to move rightwards or upwards?



4. Fill in the blanks:

$\binom{5}{0} =$ ____ $\binom{5}{1} =$ ____ $\binom{5}{2} =$ ____ $\binom{5}{3} =$ ____ $\binom{5}{4} =$ ____ $\binom{5}{5} =$ ____

5. Expand $(1+x)^5 = (1+x) \cdot (1+x) \cdot (1+x) \cdot (1+x) \cdot (1+x)$.

Some Counting Problems

6. A *valid parentheses sequence* is a list of open and closed parentheses where there are equally many open parentheses as closed parentheses, and at no point in the sequence are there more closed parentheses than open parentheses. For example, the complete list of valid parentheses sequences of length 6 is:

((())), (()()), ()()(), (())(), ()(())

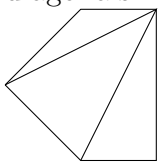
Find all valid parentheses sequences of:

Length 2:

Length 4:

Length 8:

7. A *triangulation* of a polygon is a way of dividing the polygon into triangles by drawing diagonals. Below is an example of a triangulation of a pentagon:



Find the number of possible triangulations of:

A square:

A pentagon:

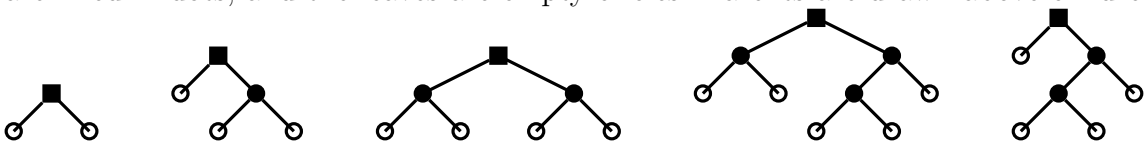
A hexagon:

A heptagon (7 sides). Try to solve this without actually drawing all of the triangulations.

8. A *rooted binary tree* is a picture with dots connected by lines. For two dots connected by a line, one is the parent and one is the child. There are three types of dots:

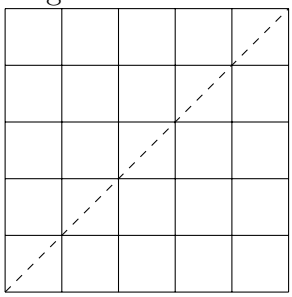
- A single **root**, which has two children (a left child and a right child).
- Branches, which have one parent and two children (a left child and a right child).
- Leaves, which have one parent and zero children.

Every dot must connect to the root. Below are some examples. The root is a square, the branches are filled in dots, and the leaves are empty circles. Parents are drawn above children.



Find the number of binary trees with 2 leaves, with 3 leaves, with 4 leaves, and with 5 leaves.

9. How many ways are there to go from the bottom left corner to the top right corner in the grid, if you have to move along the grid lines, can only move up and right, and can't go above the diagonal?



Here's one example path: