Combinatorics and Recurrence 1
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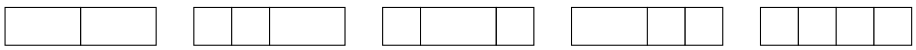
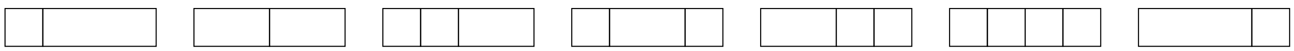
Part One: Find a recursive formula for A_n:

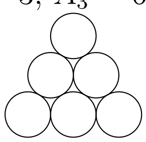
1. A_n is the number of tilings of a $1 \times n$ rectangle with blocks of size 1×1 or 1×2. Example: when $n = 4$, $A_4 = 5$:



2. A_n is the number of tilings of a $1 \times n$ rectangle with blocks of size 1×1, 1×2, or 1×3. Example: when $n = 4$, $A_4 = 7$:



3. A_n is the number of coins in a coin triangle with n coins along each side. Example: when $n = 3$, $A_3 = 6$:



4. A_n is the number of ways of choosing an arbitrary subset from the set $\{1, 2, \ldots, n\}$. Example: when $n = 3$, $A_3 = 8$:

$$\{\}, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

5. A_n is the number of permutations of $\{1, 2, \ldots, n\}$. A permutation is a way of writing the numbers in order, with each number appearing once. Example: When $n = 3$, $A_3 = 6$:

$$123, 213, 132, 231, 312, 321$$

6. A_n is the number of derangements of $\{1, 2, \ldots, n\}$. A derangement is a permutation such that no number is written in its original position. Example: When $n = 4$, $A_4 = 9$:

$$4123, 3421, 3142, 4312, 2413, 2341, 4321, 3412, 2143$$

7. (2006 AIME, #11) A collection of n cubes consists of one cube of edge length k for $1 \leq k \leq n$. A tower is to be built from all n blocks using the rules:

- Any cube may be the bottom in the tower.
- The cube immediately on top of the cube with edge length k must have edge length at most $k + 2$.

Let A_n be the number of towers that can be constructed.

8. (2007 AMC 12A, #25) A_n is the number of spacey subsets of $\{1, 2, \ldots, n\}$. A set is spacey if it contains no more than one out of any three consecutive integers.

Part Two: Find a recursive formula for A_n by introducing other sequences.

9. A_n is the number of sequences of zeroes, ones, and twos that don't have any two adjacent numbers the same.

10. Jenna and Sarah play a game, in which they flip a fair coin repeatedly. Jenna wins if the sequence HHT appears before the sequence THT, and Sarah wins otherwise. Let A_n denote the number of sequences of n flips that end with Jenna winning, meaning that THT never appears and HHT are the final three symbols.

11. (2001 AIME, #14**)** A mail carrier delivers mail to the n houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. A_n is the number of mail delivery patterns that are possible.

Part Three: Find a recursive formula for $A_{n,k}$.

12. $A_{n,k}$ is the number of ways of choosing k elements from a set of n elements.

13. $A_{n,k}$ is the number of ways of partitioning n elements into k parts.

Part Four: Find a recursive formula for A_n, that can depend on all values of A_k for $k < n$.

14. A_n is the number of triangulations of a regular $n-$gon.

15. A_n is the number of valid parentheses sequences with n open parentheses and n closed parentheses. A parentheses sequence is valid if there are never more closed parentheses than open parentheses. For example, for $n = 3$, we get ((())), (()()), ()()(), (())(), ()(()).