CONSTRUCTABILITY: STRAIGHTEDGE AND COMPASS VS. ORIGAMI

ZSUZSANNA DANCSO AND STEPHEN MORGAN (MSRI)

1. Euclidean Constructions and Constructability

Review Euclidean constructions. Euclidean constructions use two tools: a straightedge (unmarked ruler), and a compass. There are three elementary steps that you can do with the tools:

- Given two points, use the straightedge to construct the line passing through them.
- Given a point (center) and a line segment (radius), use the compass to construct a circle around the given point with the given radius.
- You can mark the points of intersection of lines and circles with each other.

Review what you can already do. You have already learned these basic Euclidean constructions:

(1) To construct the perpendicular bisector to a given line segment.
(2) To drop a perpendicular line to a given line from a given point not on the line.
(3) To erect a perpendicular line to a given line from a given point on the line.
(4) To construct a line parallel to a given line through a given point (not on the line).
(5) To construct the angle bisector to a given angle.

Review what is impossible. You also know that the following three problems *cannot* be solved by straightedge and compass.

(1) To *trisect* any given angle.
(2) To *square the circle*. (What does this mean?)
(3) To *double the cube*.
Remember that doubling the cube means that given a cube, you want to construct a cube of twice the volume. This means that the side length of the new cube has to be $\sqrt[3]{2}$ times the side length of the original cube. For example, if you are given a *unit cube*, that is, a cube of side length 1, your task would be to construct a cube with side length $\sqrt[3]{2}$, but that turns out to be impossible (this is a hard theorem that we won't prove today).

Problem. But would it be possible to double a square? Let's see:

(1) If a square has side length a, what is the side length of a square of *twice the area*?
(2) I give you a unit square (that is, the side length is 1 and the area is also 1). Can you construct a square of area 2?
(3) I give you a square of side length a. Can you double the square (i.e. construct a square of twice the area)?

Bonus problem. For those of you who thought that was too easy:

(1) Given a unit line segment (a line segment of length 1), how would you construct a line segment of length $\sqrt{3}$? Now $\sqrt{5}$? How about $\sqrt{n}$ for any positive integer n?
(2) Given a unit line segment, how would you construct a line segment of length $1/3$?
(3) Given line two segments of lengths a and b (positive real numbers, not necessarily integers), how would you construct a line segment of length $a \cdot b$? How about a/b? (Hint: use similar triangles...)

2. Origami constructions and constructibility

A different way of doing constructions is to use paper folding instead of straightedge and compass. To be able to ask any meaningful questions about what can be constructed by origami, we have to define the "axioms", or basic moves (see also the picture on the next page):

- Given any two points P and Q, you can fold a crease that goes through them.
- Given two points P and Q, you can fold P onto Q and crease the line. (What is this line in Euclidean language?)
- Given lines l and m, you can fold l onto m and mark the crease. (What does this do in Euclidean language?)
- Given a point P and a line l, you can fold a crease perpendicular to l that goes through P. (What is this in Euclid's terms?)
- Given two points P and Q and two lines l and m, you can fold P onto l and Q onto m at the same time, and mark the crease.

Question. Do you think origami is weaker or stronger than straightedge and compass? Can it accomplish more things or less? For example, is it possible that we can trisect an angle with origami moves? Double the cube? Square the circle?

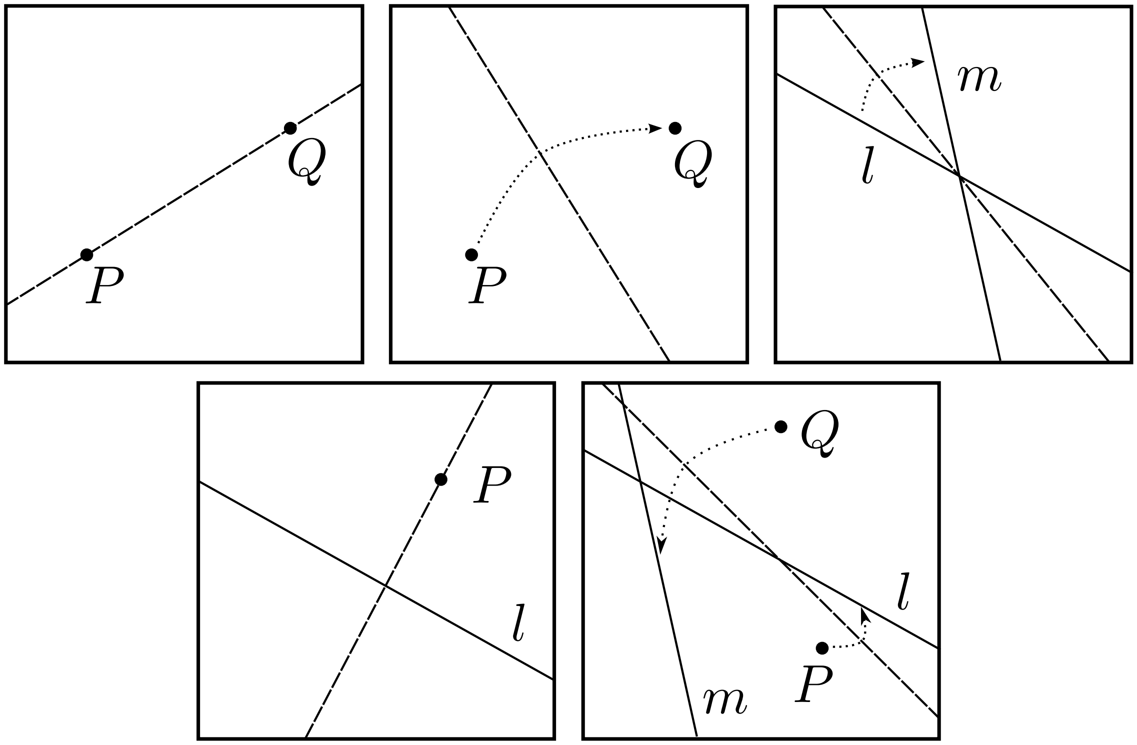
Bonus problem (homework). Use the basic origami moves to construct the following things, and compare with the Euclidean constructions of the same:

(1) A line parallel to a given line at a given point.
(2) Given a segment of length a and a non-parallel line l elsewhere with a marked point P, construct a segment of length a that lies on l and starts at P.
(3) Given a unit segment and a segment of length a, construct a segment of length $\sqrt{1 + a^2}$.

Surprise: Even though it uses no tools, origami turns out to be stronger than straightedge and compass! In particular, there is an origami construction for trisecting an angle. We'll do this together.

Problem. Prove that the origami construction for trisecting an angle indeed trisects the angle. (Hint: use congruent triangles and properties of reflection to a line)

Note. You can also double the cube with origami, and you can even use it to solve cubic equations! Squaring the circle is still impossible though (transcendental numbers are off-limits).



The five basic origami moves