4. History again: Impossible Geometry Problems

Question 1. Can all geometric construction problems be done just with a straightedge and compasses?

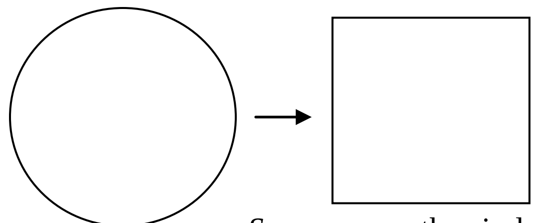
Historical Facts 1. At the heart of geometry study in antiquity lie 3 hard problems:

- circle squaring,
- cube duplication, and
- angle trisection.

The Greeks were unable to solve these problems, but it was not until hundreds of years later that the problems were proved to be actually *impossible* under the limitations imposed: to use only a straightedge and compasses.

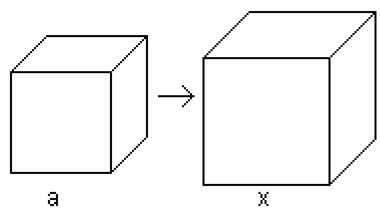
Historical Facts 2. Another famous problem is to construct a *regular* *n*-gon (polygon with *n* equal sides and *n* equal angles), using only a straightedge and a compass. You constructed the first case of this: an *equilateral triangle*. It is possible to construct a square, a regular pentagon, and a regular hexagon, but... a regular *heptagon* (7-gon) is impossible! The Greeks were able to construct regular polygons with sides 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, ... In 1796, *Gauss* (only 19 years at the time) proved that the number of sides of constructible polygons had to be of a certain form involving *Fermat primes*. He also constructed the first missing case of a regular 17-gon, 34-gon, 51-gon, and others.

Wild Goose Chase 1. (Circle Squaring) Given a circle, construct a square with area equal to that of the area of the circle, using only a straightedge and a compass.



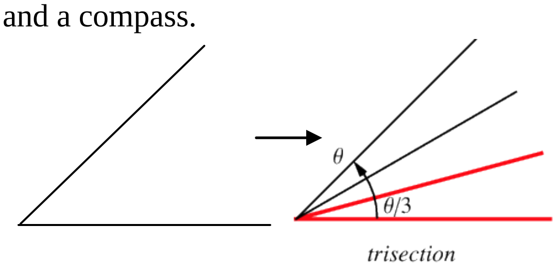
Same area as the circle?

Wild Goose Chase 2. (Cube Duplication) Given a cube, construct another cube with twice as large volume, using only a straightedge and a compass.



Twice as large volume?

Wild Goose Chase 3. (Angle Trisection) Given any angle, divide into exactly 3 equal sub-angles, using only a straightedge and a compass.



A *third* of the angle?

RECAP 1: New Vocabulary and Ideas

Check the correct answer. Explain your choice in words and provide details.

1. Euclid's *Elements* are:
   - I. A list of chemical elements found in antiquity.
   - II. A book full of geometric axioms and propositions.
   - III. A series of 13 books.
   - IV. A valuable book with antique recipes, translated into numerous languages.
2. *Euclidean constructions* are performed with:
   - I. A ruler and compasses.
   - II. A protractor and a straightedge.
   - III. Compasses and a straightedge.
   - IV. A pencil, ruler, and a needle.
3. A *straightedge* is used to:
   - I. Punish children who misbehave.
   - II. Draw a rhombus.
   - III. Measure distances.
   - IV. Find flaws in someone's drawings.
4. With a pair of *compasses* one can:
   - I. Draw ellipses.
   - II. Find his way back to Berkeley.
   - III. Construct a regular 7-gon.
   - IV. Aid others in finding the exact midpoint of a segment.
5. A *ruler* was not used by the Greek geometers because:
   - I. It is believed that they did not know how to operate with numbers written in the 10-base system.
   - II. It is rumored that they were not smart enough to use the power of the ruler.
   - III. It is known that they did not have computers to help them do arithmetic problems.
   - IV. It is suspected that they liked working only on hard and interesting geometry problems.
6. To find the exact *midpoint* of a segment one needs to:
   - I. Improve his detective abilities.
   - II. Find out how Gauss did it.
   - III. Use the properties of a special quadrilateral.
   - IV. Know that the diagonals of a rhombus are not necessarily equal in length.
7. The exact *centroid* of a triangle can be located by:
   - I. Redrawing the triangle on a graph paper and using the properties of the graph paper.
   - II. Intersecting the medians of the triangle.
   - III. Finding the approximate place of the center of mass of the triangle using an inflexible model of the triangle and hanging it on a string.
   - IV. Using a GPS device.
8. The address of the Mathematical Sciences Research Institute in Berkeley is "*17 Gauss Way*" because:
   - I. The Greeks failed to draw a regular 17-gon.
   - II. The street names in Berkeley are funny.
   - III. Gauss was 17 years old when he discovered important geometric facts.
   - IV. A regular 17-gon is the first in a sequence of polygons drawn by Gauss, using a straightedge and compasses.

9. It is *possible* to:
   I. Draw a rhombus with a ruler and compasses.
   II. Double the volume of the cube with a straightedge and compasses.
   III. Draw a regular heptagon with a straightedge and compasses.
   IV. Divide any angle into three equal sub-angles with a straightedge and compasses.

10. In Berkeley, there is a *street named*:
    I. Euclid.
    II. Archimedes.
    III. Euler.
    IV. Desire.

11. We study *Euclidean constructions* for a number of reasons EXCEPT:
    I. It is a beautiful topic that provides insights to many geometric problems.
    II. It is interesting from a historical perspective to learn about humankind.
    III. We like to make our lives really hard.
    IV. It teaches us algorithms for constructing many shapes.

12. *Euclid*:
    I. Was a famous scholar and scientist in the ancient times.
    II. Wrote one of the most famous books of all times.
    III. Used only a straightedge and compasses in the 465 propositions in his *Elements*.
    IV. All of the above.

RECAP 2: New Theory

What new facts did you learn about

- Rhombi?
- Equilateral triangles?
- Regular polygons?
- Impossible geometry problems?
- Drawing tools allowed in Euclidean constructions?

RECAP 3: Algorithms

What is an algorithm? Describe the algorithms you learnt so far, using only a straightedge and compasses, to construct:

- An equilateral triangle.
- A rhombus.
- The midpoint of any segment.
- The centroid of any triangle.

RECAP 4: Mathematical Logic

- What is the difference between a *theorem* and an *algorithm*? Which needs to be proven?
- Did we *prove* any of the new theory facts, new historical facts, or new algorithms? How are we so sure they are true? What evidence do we have that they are true?