COMBINATORICS

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Bijective proof.

1. Fifty nine teams are participating in the soccer cup. How many games will be played in this cup if we know that it is knock-out tournament?

2. An integer triangle is a triangle all of whose sides have lengths that are integers. What is the number of integer triangles, such that the sum of the shortest and longest sides is 12? And n?

3. What is the number of integer triangles such that one side is of length a and the other two sides are in proportion $\frac{p}{q}$, where p and q are relatively prime positive integers?

4. Denote by $P(n)$ the set of non-congruent integer triangles of perimeter n. Prove that if n is odd then both $P(n)$ and $P(n+3)$ contain the same number of triangles.

5. An ant is walking on the squares of a 10×10 grid. An ant path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards along the grid lines. How many paths are there if it is allowed only two turns? Three turns? If it may do the maximum number of turns?

6. A triangulation of a convex n-gon is the way to divide it into non-intersecting triangles by drawing some diagonals. Let $T(n)$ denote the number of triangulations of $n+2$-gon into n triangles. Determine $T(3)$ and $T(4)$.

7. Let $S(n)$ be the number of different ways $n+1$ summands can be completely parenthesized. Determine $S(3)$ and $S(4)$. Show that $T(n) = S(n)$.

Recursion.

8. Some stepping stones cross a small river. How many ways back to the bank are there if you are standing on the n-th stone? You can either step on

to the next stone or hop over one stone to land on the next. How many ways back to the bank are there if you can not hop over a stone twice in a row?

9. Consider a set of n lines, lying in a plane, no three of them concurrent in a point and no two are parallel. What is the number of regions into which these lines divide the plane?

10. What is the maximum number of regions into which n circles can divide the plane? How are they arranged?

Binomial coefficients.

11. For any $n \geq k \geq 1$ prove the identities

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \qquad \binom{n}{k} = \binom{n}{n-k}$$

12. Let $m \times n$ be a rectangular grid and define path in the same way as in Problem 5. How many paths are between $(0,0)$ and (m,n)?

13. Find a combinatorial model to prove that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

14. Are there consecutive binomial coefficients $\binom{n}{k-1}$, $\binom{n}{k}$ and $\binom{n}{k+1}$ that form a geometric progression?

Catalan numbers.

15. Let $n \times n$ be a square grid. How many paths are there which do not pass above the diagonal between $(0,0)$ and (n,n)?

16. *(Chung-Feller Theorem)* Let $n \times n$ be a square grid. We set $K(i)$, $0 \leq i \leq n$, be the set of all paths which have exactly i edges over diagonal between $(0,0)$ and (n,n). Show that all $K(i)$'s contain the same number of elements.

17. *("Catalania"=Catalan mania)* Check 66 combinatorial interpretations of Catalan numbers in the book of Richard P. Stanley, "Enumerative Combinatorics", V.2 p.219 and in http://www-math.mit.edu/~ rstan/ec/catadd.pdf