Berkeley Math Circle
Monthly Contest 8, Solutions

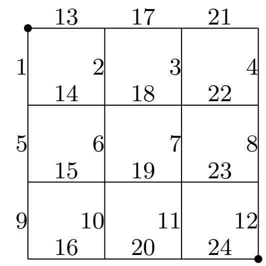
1. Find a positive integer satisfying the following three clues:

   - When multiplied by 2, it yields a perfect square.
   - When multiplied by 3, it yields a perfect cube.
   - When multiplied by 5, it yields a perfect fifth power.

   Solution. It is reasonable to try a number of the form $2^a3^b5^c$, where a, b, and c are nonnegative integers. We want

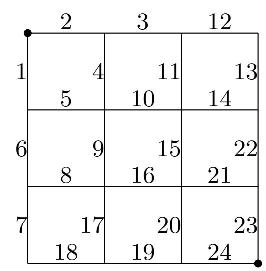
$$\begin{array}{llll} 2^{a+1}3^b5^c \text{ is a square, i.e.} & 2|a+1, & 2|b, & \text{and } 2|c \\ 2^a3^{b+1}5^c \text{ is a cube, i.e.} & 3|a, & 3|b+1, & \text{and } 3|c \\ 2^a3^b5^{c+1} \text{ is a fifth power, i.e.} & 5|a, & 5|b, & \text{and } 5|c+1. \end{array}$$

   We can now search for a, b, and c separately. We find that $n_0 = 2^{15}3^{20}5^{24}$ is a solution.



2. Determine whether it is possible to write the numbers $1, 2, \ldots, 24$ on the edges of the 3×3 grid of squares shown, one number to each edge, such that the sum of the six numbers on every path of minimal length from the upper left corner to the lower right corner is the same.

   Solution. It is possible, and two especially simple solutions are shown. To check that they work without going through every path, one can use the following strategy: First notice that in each of the nine unit squares, the sum of the left and bottom edges equals the sum of the top and right edges. Now, given any path that includes a rightward move (R) followed by a downward move (D), we can replace RD by DR without changing the sum of the edges. Now a minimal path consists of three R's and three D's, and each D can move past each R at most once; so after at most nine steps the path will be converted to DDDRRR. This proves that all paths have the same sum.



3. Let $A_1A_2\cdots A_{2n}$ be a convex $2n$-gon. Prove that there is an i ($1 \le i \le n$) and a pair of parallel lines, each intersecting the $2n$-gon only once, one at A_i and one at A_{n+i}.

   Solution. Consider the "highest" and "lowest" points of the $2n$-gon, with respect to any chosen orientation. Unless a side of the $2n$-gon is perfectly horizontal, these points will be unique and thus a horizontal line through them will not meet the $2n$-gon again. These two points will also be vertices of the $2n$-gon; call them A_i and A_j respectively. Say that the *imbalance* in this situation is $i - j \bmod 2n$, which (being nonzero) we can think of as an integer from 1 to $2n - 1$ inclusive. We would like to prove that the $2n$-gon can be rotated such that the imbalance is n.

   Consider smoothly rotating the $2n$-gon counterclockwise (assume that the vertices are labeled $A_1, \ldots, A_{2n}$ in counterclockwise order). Note that the imbalance does not change unless one of the sides becomes momentarily horizontal. If the top side becomes horizontal, it goes down by 1; if the bottom side becomes horizontal, it goes up by 1; and if both sides momentarily become horizontal, it does not change at all. But after rotating by 180°, the top and bottom vertices are switched: if the imbalance was between 0 and n, it is now between n and $2n$ and vice versa. Since the imbalance can never be 0 and never jumps by more than 1, it must be n at some point.

4. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ such that

$$f(x)f(x+1) = g(h(x)).$$

   Solution. Write $f(x) = ax^2 + bx + c$, $a \neq 0$. It is a familiar fact that the graph of a quadratic function always has an axis of symmetry, specifically the line $x = -\frac{b}{2a}$. By substituting

$$u = x + \frac{b}{2a} + \frac{1}{2},$$

we can make $f(x) = au^2 - au + d$ symmetric about the line $u = 1/2$. Then $f(x+1) = au^2 + au + d$ is symmetric about the line $u = -1/2$, and their product is symmetric about the line $u = 0$:

$$\begin{aligned} f(x)f(x+1) &= (au^2 - au + d)(au^2 + au + d) \\ &= a^2u^4 + 2adu^2 - a^2u^2 + d^2 \\ &= g(u^2), \end{aligned}$$

where $g(v) = a^2v^2 + (2ad - a^2)v + d^2$. But now

$$u^2 = \left(x + \frac{b}{2a} + \frac{1}{2}\right)^2$$

is a quadratic function $h(x)$ of x.

5. For x a positive real number with finitely many decimal places, denote by $r(x)$ the number formed by reversing the digits and decimal point of x. For instance, $r(98.6) = 6.89$ and $r(740) = 0.047$.

   (a) Prove that for all positive real numbers x and y with finitely many decimal places,

   $$r(xy) \le 10r(x)r(y).$$

   (b) Determine whether there exist values of x and y, each having at least 2015 nonzero digits, such that equality holds.

Solution. (a) Write

$$x = \sum_{i=i_0}^{i_1} x_i 10^i \quad \text{and} \quad y = \sum_{j=j_0}^{j_1} y_j 10^j,$$

where the x_i and y_j are digits, and the limits i_0, i_1, j_0, j_1 are integers, possibly negative. Then

$$r(x) = \sum_{i=i_0}^{i_1} x_i 10^{-1-i} \quad \text{and} \quad y = \sum_{j=j_0}^{j_1} y_j 10^{-1-j},$$

and we may multiply:

$$xy = \sum_{i,j} x_i y_j 10^{i+j} \quad \text{and} \quad r(x)r(y) = \sum_{i,j} x_i y_j 10^{-2-i-j}.$$

Let (initially) $z_k = \sum_{i+j=k} x_i y_j$, a finite sum since x has only finitely many nonzero digits. Then

$$A := \sum_k z_k 10^k = xy, \quad \text{and} \quad B := \sum_k z_k 10^{-1-k} = 10r(x)r(y).$$

If all $z_k < 10$, then the z_k are the digits of A and we simply get $B = r(A)$. However, in general, there will be carrying involved in the multiplication of x by y. Consider the following "carry transform": find the least k such that $z_k \ge 10$, decrease z_k by 10, and increase z_{k+1} by 1. This does not change the sum A at all, and it decreases the sum B by $99 \cdot 10^{-2-k}$. We can only apply the carry transform finitely many times since, once z_k is brought below 10, we move on to z_{k+1}, and continue increasing k until $10^k > A$, at which point z_k must remain zero. Now A still equals xy, and the z_k are its digits; and B has been decreasing so

$$r(xy) = r(A) = B \le 10r(x)r(y).$$

(b) The answer is yes; the choice

$$x = \overbrace{111\cdots1}^{2015}, y = \overbrace{1\overbrace{000\cdots0}^{2014}1\overbrace{000\cdots0}^{2014}1\cdots1}^{2015\text{ 1's}}$$

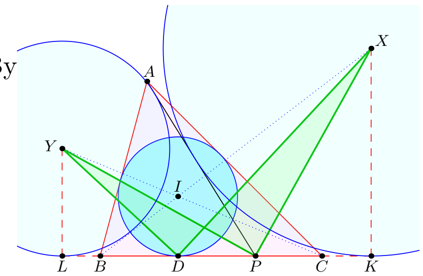
is readily seen to yield

$$r(xy) = 10r(x)r(y) = 0.\overbrace{11111\cdots1}^{2015^2}.$$

6. Let ABC be an acute triangle, and P a point on the interior of side BC. Let I be the incenter of triangle ABC, and denote by D the foot of the altitude from I to BC. Line BI meets the internal angle bisector of $\angle APC$ at X, while line CI meets the internal angle bisector of $\angle APB$ at Y. Show that the points D, P, X, Y lie on a circle.

Solution. First, notice that $\angle YPX$ is a right angle. Thus, we claim that in fact D and P lie on a circle with diameter $\overline{XY}$; in light of this it suffices to prove that

$$DX^2 + DY^2 = PX^2 + PY^2.$$



Let K and L be the feet of the altitudes from X and Y to $\overline{BC}$, respectively. By the Pythagorean Theorem we may write these as

$$\begin{aligned} DX^2 &= DK^2 + XK^2 \\ DY^2 &= DL^2 + YL^2 \\ PX^2 &= PK^2 + XK^2 \\ PY^2 &= PL^2 + YL^2 \end{aligned}$$

In light of this it is sufficient to show that

$$DK^2 + DL^2 = PK^2 + PL^2.$$

Without loss of generality, assume L, D, P, K lie in that order. Note that X is the B-excenter of $\triangle ABP$. From this configuration we derive that

$$\begin{aligned} BK &= \frac{1}{2}(AB + AP + BP) \\ PK &= \frac{1}{2}(AB + AP - BP) \end{aligned}$$

From this we derive that

$$\begin{aligned} DK + PK &= BK + PK - BD \\ &= \frac{1}{2}(2AB + 2AP - (AB + BC - AC)) \\ &= \frac{1}{2}(2AP + AB + AC - BC) \end{aligned}$$

Repeating the same calculation with $\triangle ACP$, we see this equals $DL + PL$ too. Thus, we have

$$DK + PK = DL + PL.$$

Multiplying both sides by $DP = DK - PK = PL - DL$, we obtain $DK^2 - PK^2 = PL^2 - DL^2$, completing the proof.

7. A *fissile square* is a positive integer which is a perfect square, and whose digits form two perfect squares in a row, which will be called the *left square* and the *right square*. For example, 49, 1444, and 1681 are fissile squares (decomposing as 4|9, 144|4, and 16|81 respectively). Neither the left square nor the right square may begin with the digit 0.

   (a) Prove that every square with an even number of digits is the right square of only finitely many fissile squares.

   (b) Prove that every square with an odd number of digits is the right square of infinitely many fissile squares.

Solution. We denote the right square by r^2, its number of digits by d, the left square by y^2, and the entire fissile square by x^2, so

$$x^2 = 10^d \cdot y^2 + r^2.$$

(a) If $d = 2k$ is even, then

$$\begin{aligned} r^2 &= x^2 - 10^{2k} \cdot y^2 \\ &= (x + 10^k y)(x - 10^k y). \end{aligned}$$

Now both factors are positive integers, at most r^2, so their average x is also at most r^2. This shows that the fissile square x^2 can only take on finitely many values.

(b) We first show that 1 is the right square of infinitely many fissile squares, i.e. that

$$x^2 - 10y^2 = 1$$

has infinitely many solutions. Equations of the form $x^2 - Dy^2 = \pm 1$ are called *Pell equations* and have been studied in great detail; but here the solution is not so difficult. First note that 361 is one such fissile square: there is a solution $x = 19$, $y = 6$. Next, consider the positive integers x_n, y_n that arise from expanding

$$(19 + 6\sqrt{10})^n = x_n + y_n\sqrt{10}.$$

They may also be described by the recursion $x_0 = 1, y_0 = 0, x_{n+1} = 60x_n + 19y_n, y_{n+1} = 19y_n + 6x_n$; they also satisfy the conjugate relation

$$(19 - 6\sqrt{10})^n = x_n - y_n\sqrt{10}.$$

Then

$$x_n^2 - 10y_n^2 = (x_n + y_n\sqrt{10})(x_n - y_n\sqrt{10}) = [(19 + 6\sqrt{10})(19 - 6\sqrt{10})]^n = 1^n = 1.$$

Also the x_n are strictly increasing, so the x_n^2 provide infinitely many fissile squares with right square 1.

We now turn to the general case, where r^2 has $d = 2k + 1$ digits. We first find values of n for which $10^k | y_n$. Note that when the pairs (x_n, y_n) are reduced mod y_n, there are only finitely many possibilities, so there exist $m \geq 0$, $n \geq 1$ such that

$$x_{m+n} \equiv x_m \quad \text{and} \quad y_{m+n} \equiv y_m \pmod{10^k}.$$

We can now run the recursion backwards, using the relations $x_{n-1} = -60x_n + 19y_n, y_{n+1} = 19y_n - 6x_n$ that describe multiplication by $19 - 6\sqrt{10}$, to deduce that

$$x_n \equiv x_0 = 1 \quad \text{and} \quad y_n \equiv y_0 = 0 \pmod{10^k}.$$

So $10^k | y_n$, and moreover the recursion becomes periodic at the nth step so $y_{2n}, y_{3n}, \ldots$ are divisible by 10^k. We can now multiply each of the corresponding fissile squares

$$(\text{square})\,\overbrace{0\cdots0}^{2k}1$$

by r^2 to get fissile squares

$$(\text{square})r^2$$

with right square r^2.