Berkeley Math Circle
Monthly Contest 8
Due May 5, 2015

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 8, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find a positive integer satisfying the following three clues:

   - When multiplied by 2, it yields a perfect square.

   - When multiplied by 3, it yields a perfect cube.

   - When multiplied by 5, it yields a perfect fifth power.

   Remark. It is not necessary to write down the digits of the number, only to give some kind of formula or description that shows that it exists.

2. Determine whether it is possible to write the numbers $1, 2, \ldots, 24$ on the edges of the 3×3 grid of squares shown, one number to each edge, such that the sum of the six numbers on every path of minimal length from the upper left corner to the lower right corner is the same. For example, the lower diagram shows one way to do this for a 1×2 grid.

3. Let $A_1 A_2 \cdots A_{2n}$ be a convex $2n$-gon. Prove that there is an i ($1 \leq i \leq n$) and a pair of parallel lines, each intersecting the $2n$-gon only once, one at A_i and one at A_{n+i}.

4. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ such that

$$f(x)f(x+1) = g(h(x)).$$

5. For x a positive real number with finitely many decimal places, denote by $r(x)$ the number formed by reversing the digits and decimal point of x. For instance, $r(98.6) = 6.89$ and $r(740) = 0.047$.

   (a) Prove that for all positive real numbers x and y with finitely many decimal places,

   $$r(xy) \leq 10r(x)r(y).$$
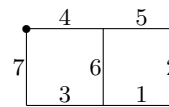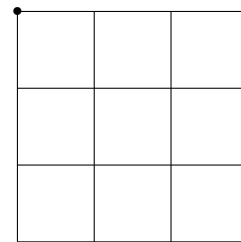
(b) Determine whether there exist values of x and y, each having at least 2015 nonzero digits, such that equality holds.

6. Let ABC be an acute triangle, and P a point on the interior of side BC. Let I be the incenter of triangle ABC, and denote by D the foot of the altitude from I to BC. Line BI meets the internal angle bisector of $\angle APC$ at X, while line CI meets the internal angle bisector of $\angle APB$ at Y. Show that the points D, P, X, Y lie on a circle.

7. A *fissile square* is a positive integer which is a perfect square, and whose digits form two perfect squares in a row, which will be called the *left square* and the *right square*. For example, 49, 1444, and 1681 are fissile squares (decomposing as 4|9, 144|4, and 16|81 respectively). Neither the left square nor the right square may begin with the digit 0.

   (a) Prove that every square with an even number of digits is the right square of only finitely many fissile squares.

   (b) Prove that every square with an odd number of digits is the right square of infinitely many fissile squares.