Berkeley Math Circle
Monthly Contest 5
Due February 3, 2015

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 4, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Determine the number of ways to fill a 3×3 grid with 0's and 1's such that each row and column has an even sum.

2. Prove that there exists a polynomial $f(x, y, z)$ with the following property: the numbers $|x|$, $|y|$, and $|z|$ are the sides of a triangle if and only if $f(x, y, z) > 0$.

   Remark. A *triangle* has three noncollinear points as vertices; thus $1, 2, 3$ or $0, 1, 1$ cannot be the sides of a triangle, but $1, 2, 2$ can.

   A *polynomial* is simply any expression that can be built out of x, y, z, and real numbers using addition, subtraction, and multiplication; thus $3x^2yz - (x + \pi y)^9$ is a polynomial.

3. A *triangular number* is one of the numbers $1, 3, 6, 10, 15, \ldots$ of the form $T_n = 1 + 2 + \cdots + n$ or, equivalently, $T_n = (n^2 + n)/2$.

   Find, with proof, all ways of writing 2015 as the difference of two triangular numbers.

4. Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and intersect at P. Let h_1, h_2, h_3, h_4 be the lengths of the altitudes from P to AB, BC, CD, DA. Show that
$$\frac{1}{h_1^2} + \frac{1}{h_3^2} = \frac{1}{h_2^2} + \frac{1}{h_4^2}.$$

5. Weighted coins numbered $2, 3, 4, \ldots, 2015$ are tossed. The coin numbered i comes up heads with probability $1/(2i^2)$. What is the probability that an odd number of coins come up heads?

6. Triangle ABC has circumcircle Γ. A circle with center O is tangent to BC at P and internally to Γ at Q, so that Q lies on arc BC of Γ not containing A. Prove that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

7. Determine if there exist positive integers a, b, m, n such that $a \neq b$, $m \geq 2$, $n \geq 2$, and
$$\underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{m} = \underbrace{b^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n}.$$