Berkeley Math Circle
Monthly Contest 4
Due January 13, 2015

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 4, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A toy slot machine accepts two kinds of coins: red and green. When a coin is inserted, the machine returns 5 coins of the other color. Laura starts with one green coin. Can it happen that after a while, she has the same number of coins of each color?

2. Find the largest number n having the following properties:

   (a) No two digits of n are equal.

   (b) The number formed by reversing the digits of n is divisible by 8.

   Remark. n cannot start with 0, but it can end with 0.

3. Determine all integers n for which $n^2 + 15$ is the square of an integer.

   Remark. Because the problem asks you to "determine all integers n", you must verify that all the n you find have the desired property, and moreover prove that these are the only such integers n.

4. Let ABC be a triangle. The incircle, centered at I, touches side BC at D. Let E be the reflection of D through I, and let F be the reflection of D through the midpoint of BC. Prove that A, E, and F are collinear.

5. A *strip* of width ℓ is the set of all points which lie on, or between, two parallel lines which are a distance ℓ apart. Let $\mathcal{S}$ be a set of 2015 points on the plane such that any three different points of $\mathcal{S}$ can be covered by a strip of width 1. Prove that $\mathcal{S}$ can be covered by a strip of width 2.

6. Show that the polynomial $(x^2 + x)^{2^{1000}} + 1$ cannot be factored as the product of two nonconstant polynomials with integer coefficients.

7. Find all nonnegative integer solutions (a, b, c, d) to the equation

$$2^a 3^b - 5^c 7^d = 1.$$